IYGB – FP1 PAPER R – QUESTION 1

METHOD A – USING RELATIONSHIPS OF ROOTS

$$x^3 - 2x^2 - 8x + 11 = 0$$

- $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-2}{1} = 2$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{-8}{1} = -8$
- $\alpha\beta\gamma = -\frac{d}{a} = \frac{-11}{1} = -11$

PROCEED AS FOLLOWS

$$A = \alpha + 1 \quad , \quad B = \beta + 1 \quad , \quad C = \gamma + 1$$

- $A + B + C = (\alpha+1) + (\beta+1) + (\gamma+1) = (\alpha+\beta+\gamma) + 3$

$$= 2 + 3 = 5$$

- $AB + BC + CA = (\alpha+1)(\beta+1) + (\beta+1)(\gamma+1) + (\gamma+1)(\alpha+1)$

$$= \alpha\beta + \alpha + \beta + 1$$
$$\beta\gamma + \beta + \gamma + 1$$
$$\alpha\gamma + \alpha + \gamma + 1$$
$$= (\alpha\beta + \beta\gamma + \gamma\alpha) + 2(\alpha+\beta+\gamma) + 3$$
$$= -8 + (2 \times 2) + 3$$
$$= -1$$

- $ABC = (\alpha+1)(\beta+1)(\gamma+1) = (\alpha+1)(\beta\gamma + \beta + \gamma + 1)$

$$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1$$
$$= \alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha+\beta+\gamma) + 1$$
$$= -11 - 8 + 2 + 1$$
$$= -16$$

IYGB – FP1 PAPER R – QUESTION 1

HENCE THE REQUIRED EQUATION WILL BE

$$x^3 - (A+B+C)x^2 + (AB+BC+CA)x - (ABC) = 0$$

$$x^3 - 5x^2 - x - (-16) = 0$$

$$x^3 - 5x^2 - x + 16 = 0$$

METHOD B – SOLUTION BY "FORCING"

LET $y = x+1 \implies x = y-1$

SUBSTITUTE INTO THE CUBIC

$$\implies x^3 - 2x^2 - 8x + 11 = 0$$

$$\implies (y-1)^3 - 2(y-1)^2 - 8(y-1) + 11 = 0$$

$$\implies y^3 - 3y^2 + 3y - 1 - 2(y^2 - 2y + 1) - 8y + 8 + 11 = 0$$

$$\implies \begin{aligned} y^3 - 3y^2 + 3y - 1 & \\ - 2y^2 + 4y - 2 & \\ - 8y + 19 & = 0 \end{aligned}$$

$$\implies y^2 - 5y^2 - y + 16 = 0$$

AS BEFORE

FP1 - PAPER R - QUESTION 2

PROCEED AS FOLLOWS — AS QUARTIC HAS REAL COEFFICIENTS ANY COMPLEX ROOTS WILL APPEAR AS CONJUGATE PAIRS

SO $\quad z_1 = 2 \qquad z_2 = 1+2i \qquad z_3 = 1-2i$

NOW THE SUM OF ALL 4 ROOTS SATISFY

$$z_1 + z_2 + z_3 + z_4 = \text{"}-\frac{b}{a}\text{"}$$
$$2 + (1+2i) + (1-2i) + z_4 = -\frac{-3}{1}$$
$$4 + z_4 = 3$$
$$z_4 = -1$$

THUS WE HAVE

$$\Rightarrow \left[z-(1+2i)\right]\left[z-(1-2i)\right](z+1)(z-2) = 0$$
$$\Rightarrow \left[(z-1)-2i\right]\left[(z-1)+2i\right](z^2-z-2) = 0$$
$$\Rightarrow \left[(z-1)^2-(2i)^2\right](z^2-z-2) = 0$$
$$\Rightarrow \left[z^2-2z+1-(-4)\right](z^2-z-2) = 0$$
$$\Rightarrow (z^2-2z+5)(z^2-z-2) = 0$$
$$\Rightarrow z^4 - z^3 - 2z^2 - 2z^3 + 2z^2 + 4z + 5z^2 - 5z - 10 = 0$$
$$\Rightarrow z^4 - 3z^3 + 5z^2 - z - 10 = 0$$

$\therefore \quad a = 5$

$b = -1$

$c = -10$

IYGB - FP1 PAPER R - QUESTION 3

a) BY VERIFICATION

$\vec{AB} = \underline{b} - \underline{a} = (0,2,2) - (3,1,0) = (-3,1,2)$

$\vec{AC} = \underline{c} - \underline{a} = (3,3,1) - (3,1,0) = (0,2,1)$

DOTTING EACH OF THESE VECTORS WITH THE NORMAL GIVEN

$(-3,1,2) \cdot (1,-1,2) = -3 - 1 + 4 = 0$

$(0,2,1) \cdot (1,-1,2) = 0 - 2 + 2 = 0$

INDEED THE NORMAL TO Π //

b) THE EQUATION OF THE PLANE WILL BE

$x - y + 2z = \text{CONSTANT}$

USING ANY OF THE 3 POINTS, SAY $B(0,2,2)$

$0 - 2 + 2 \times 2 = \text{CONSTANT}$

$\text{CONSTANT} = 2$

$\therefore \; x - y + 2z = 2$ //

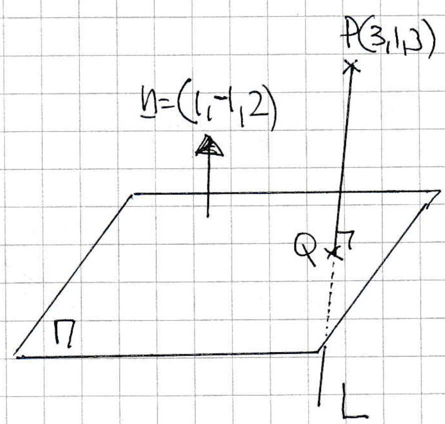
c) STANDARD APPROACH

$L: \; \underline{r} = (3,1,3) + \lambda(1,-1,2)$

$\underline{r} = (\lambda+3, \; 1-\lambda, \; 2\lambda+3)$

$\uparrow \quad \uparrow \quad \uparrow$

$x \quad y \quad z$



SOLVING SIMULTANEOUSLY WITH Π

$\Rightarrow x - y + 2z = 2$

$\Rightarrow (\lambda+3) - (1-\lambda) + 2(2\lambda+3) = 2$

IYGB – FP1 PAPER R – QUESTION 3

$\Rightarrow \lambda + 3 - 1 + \lambda + 4\lambda + 6 = 2$

$\Rightarrow 6\lambda + 8 = 2$

$\Rightarrow 6\lambda = -6$

$\Rightarrow \lambda = -1$

$\therefore Q(\underset{\lambda+3}{2}, \underset{1-\lambda}{2}, \underset{2\lambda+3}{1})$

$|PQ| = |q - p| = |(2,2,1) - (3,1,3)| = |-1,1,-2| = \sqrt{1+1+4}$

$\therefore |PQ| = \sqrt{6}$

ALTERNATIVE FOR PART (c)

- $\overrightarrow{PA} = a - p = (3,1,0) - (3,1,3) = (0,0,-3)$
- $\hat{n} = \frac{1}{\sqrt{1+1+4}}(1,-1,2) = \frac{1}{\sqrt{6}}(1,-1,2)$
- $|PQ| = |\overrightarrow{PA} \cdot \hat{n}|$

$= \left|(0,0,-3) \cdot \frac{1}{\sqrt{6}}(1,-1,2)\right|$

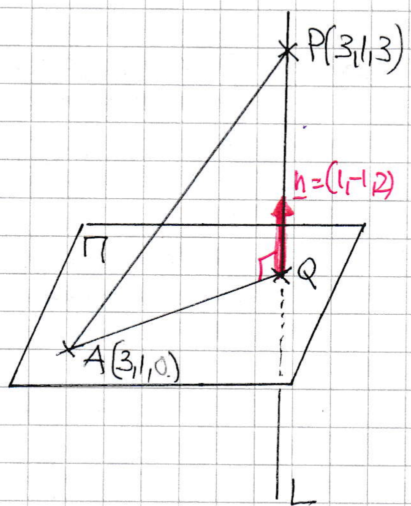
$= \frac{1}{\sqrt{6}}\left|(0,0,-3) \cdot (1,-1,2)\right|$

$= \frac{1}{\sqrt{6}}|0 + 0 - 6|$

$= \frac{6}{\sqrt{6}}$

$= \sqrt{6}$

AS BEFORE



IYGB - FP1 PAPER R - QUESTION 4

MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{x}{1+i} = \frac{1-5i}{3-2i} + \frac{y}{2-i}$$

$$\Rightarrow \frac{x(1-i)}{(1+i)(1-i)} = \frac{(1-5i)(3+2i)}{(3-2i)(3+2i)} + \frac{y(2+i)}{(2-i)(2+i)}$$

$$\Rightarrow \frac{x(1-i)}{2} = \frac{3+2i-15i-10i^2}{9+4} + \frac{y(2+i)}{4+1}$$

$$\Rightarrow \frac{x(1-i)}{2} = \frac{13-13i}{13} + \frac{y(2+i)}{5}$$

$$\Rightarrow \frac{x(1-i)}{2} = 1-i + \frac{y(2+i)}{5}$$

$$\Rightarrow 5x(1-i) = 10-10i + 2y(2+i)$$

$$\Rightarrow 5x - 5xi = 10 - 10i + 4y + 2yi$$

$$\Rightarrow 5x - 5xi = (10+4y) + (2y-10)i$$

EQUATING REAL & IMAGINARY PARTS

$$\left.\begin{array}{l} 5x = 10 + 4y \\ -5x = 2y - 10 \end{array}\right\} \text{ADD} \Rightarrow 0 = 6y$$

$$\Rightarrow \underline{\underline{y = 0}}$$

& HENCE $\underline{\underline{x = 2}}$

IYGB - FP1 PAPER R - QUESTION 5

START BY WRITTING THE SERIES IN SIGMA NOTATION

$$\underbrace{(1\cdot2\cdot3)+(2\cdot3\cdot5)+(3\cdot4\cdot7)+(4\cdot5\cdot9)+\dots}_{\text{N TERMS}} = \sum_{r=1}^{n}\left[r(r+1)(2r+1)\right]$$

EXPAND & SIMPLIFY

$$\sum_{r=1}^{n}\left[r(r+1)(2r+1)\right] = \sum_{r=1}^{n}\left[2r^3+3r^2+r\right]$$

$$= 2\sum_{r=1}^{n}r^3 + 3\sum_{r=1}^{n}r^2 + \sum_{r=1}^{n}r$$

USING STANDARD RESULT

$$= 2\times\frac{1}{4}n^2(n+1)^2 + 3\times\frac{1}{6}\times n(n+1)(2n+1) + \frac{1}{2}\times n(n+1)$$

$$= \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{2}n(n+1)\left[n(n+1) + (2n+1) + 1\right]$$

$$= \frac{1}{2}n(n+1)\left[n^2+3n+2\right]$$

$$= \frac{1}{2}n(n+1)(n+1)(n+2)$$

$$= \frac{1}{2}n(n+1)^2(n+2)$$

IYGB - FP1 PAPER R - QUESTION 6

START BY PARAMETRIZING THE LINE FROM CARTESIAN

$$\frac{x-2}{3} = \frac{y+2}{4} = \frac{1-z}{2} = t$$

$$x = 3t+2$$
$$y = 4t-2$$
$$z = 1-2t$$

APPLY THE TRANSFORMATION IN GENERAL FORM

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3t+2 \\ 4t-2 \\ 1-2t \end{bmatrix} = \begin{bmatrix} 3t+2+1-2t \\ 6t+4+4t-2+1-2t \\ 1-2t \end{bmatrix} = \begin{bmatrix} t+3 \\ 8t+3 \\ 1-2t \end{bmatrix}$$

ELIMINATE THE PARAMETER

$$t = X-3$$
$$t = \frac{Y-3}{8}$$
$$t = \frac{1-Z}{2}$$

$$\therefore \; X-3 = \frac{Y-3}{8} = \frac{1-Z}{2}$$

OR

$$x-3 = \frac{y-3}{8} = \frac{1-z}{2}$$

IYGB – FP1 PAPER R – QUESTION 7

a) WRITE THE EQUATIONS IN PARAMETRIC FORM

$$\underline{r}_1 = (12,7,3) + \lambda(2,1,1) = (2\lambda+12, \lambda+7, \lambda+3)$$
$$\underline{r}_2 = (1,3,0) + \mu(3,0,-1) = (3\mu+1, 3, -\mu)$$

EQUATE $\underline{j}$ & $\underline{k}$

$$\left.\begin{array}{l} \underline{j}: \lambda+7=3 \\ \underline{k}: \lambda+3=-\mu \end{array}\right\} \Rightarrow \lambda=-4 \quad \& \quad \mu=1$$

CHECK $\underline{i}$

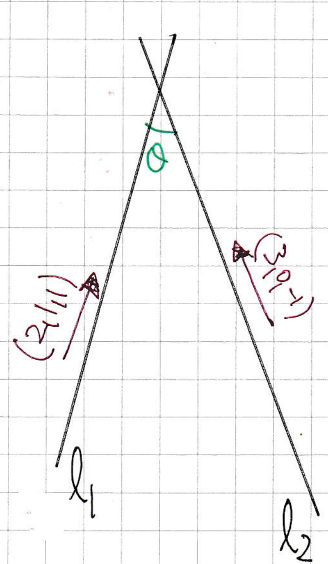
$$2\lambda+12 = 2(-4)+12 = 4$$
$$3\mu+1 = 3\times1+1 = 4$$

AS ALL 3 COMPONENTS AGREE WHEN $\lambda=-4$ & $\mu=1$ THE LINES INTERSECT AT SOME POINT

USING $\lambda=-4$ OR $\mu=1$ YIELDS $A(4,3,-1)$

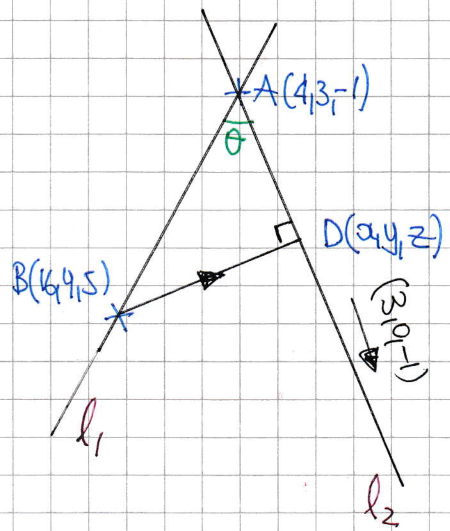
b) DOTTING THE DIRECTION VECTORS

$$(2,1,1)\cdot(3,0,-1) = |2,1,1|\times|3,0,-1|\cos\theta$$
$$6+0-1 = \sqrt{4+1+1}\,\sqrt{9+0+1}\,\cos\theta$$
$$5 = \sqrt{6}\,\sqrt{10}\,\cos\theta$$
$$\cos\theta = \frac{5}{\sqrt{60}}$$
$$\theta \simeq 49.8°$$



IYGB – FP1 PAPER R QUESTION 7

c) LOOKING AT THE DIAGRAM



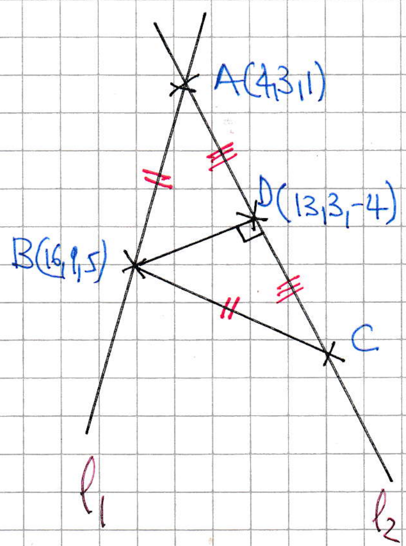
- $\overrightarrow{BD} = \underline{d} - \underline{b} = (x,y,z) - (16,9,5)$
  $= (x-16,\ y-9,\ z-5)$
- $(x-16,\ y-9,\ z-5) \cdot (3,0,-1) = 0$
  (AS $B\hat{D}A = 90°$)

  $3x - 48 - z + 5 = 0$
  $3x - z = 43$
- BUT $(x,y,z) = (3\mu+1,\ 3,\ -\mu)$

  $\Longrightarrow 3(3\mu+1) - (-\mu) = 43$
  $\Longrightarrow 10\mu + 3 = 43$
  $\Longrightarrow 10\mu = 40$
  $\Longrightarrow \mu = 4$

  $\therefore D(13,3,-4)$

d) THE POINT C IS NOT UNIQUE, HOWEVER C COULD BE SUCH SO THAT D IS THE MIDPOINT OF AC



A		D		C
4	$\xrightarrow{+9}$	13	$\xrightarrow{+9}$	22
3	$\xrightarrow{+0}$	3	$\xrightarrow{+0}$	3
-1	$\xrightarrow{-3}$	-4	$\xrightarrow{-3}$	-7

$\therefore C(22,3,-7)$

IYGB - FP1 PAPER R - QUESTION 8

$$\sum_{r=1}^{n} \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geqslant 1$$

BASE CASE; n=1

- $\text{LHS} = \dfrac{2\times1^2-1}{1^2(1+1)^2} = \dfrac{1}{4}$
- $\text{RHS} = \dfrac{1^2}{(1+1)^2} = \dfrac{1}{4}$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT, HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\Rightarrow \sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$$

$$\Rightarrow \sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)} + \frac{2(k+1)^2-1}{(k+1)^2(k+1+1)^2} = \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^2(k+2)^2 + 2(k+1)^2-1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^2(k+2)^2 + 2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^4+4k^3+4k^2+2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^4+4k^3+6k^2+4k+1}{(k+1)^2(k+2)^2}$$

IYGB - FP1 PAPER R - QUESTION 8

BY INSPECTION

$$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1$$

RETURNING TO THE MAIN LINE

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k+1}{((k+1)+1)^2}$$

CONCLUSION

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL n