Worksheet 10 Solutions

Question 1 Solution.

Before we can differentiate y expression, we need to simplify it:

$$
\begin{aligned}
y &= \frac{x^3 - 100x}{x^2 + 10x} \\
&= \frac{\cancel{x}(x^2 - 100)}{\cancel{x}(x + 10)} \\
&= \frac{\cancel{(x + 10)}(x - 10)}{\cancel{(x + 10)}} \\
&= x - 10
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}(x - 10)\right) \\
&= \frac{d}{dx}(1) \\
&= 0
\end{aligned}
$$

as required.

Question 2 Solution.

(a) Completing the square gives:

$$\begin{aligned}
&3x^2 + 3y^2 - 2x + 6y = 9 \\
&\Rightarrow x^2 + y^2 - \frac{2}{3}x + 2y = 3 \\
&\Rightarrow \left(x - \frac{1}{3}\right)^2 - \frac{1}{9} + (y+1)^2 - 1 = 3 \\
&\Rightarrow \left(x - \frac{1}{3}\right)^2 + (y+1)^2 = \frac{37}{9}
\end{aligned}$$

so the centre of the circle is $\left(\frac{1}{3}, -1\right)$ and the radius of the circle is $\frac{\sqrt{37}}{3}$.

(b)

$$3(0)^2 + 3(1)^2 - 2(0) + 6(1) = 9$$

so the point $P(0,1)$ satisifies the equation of the circle C. Hence, P lies on the circle.

The normal to C at P has gradient

$$\frac{1 - -1}{0 - \frac{1}{3}} = -6$$

and so the tangent to C at P has gradient $\frac{1}{6}$.

Hence, the equation of the tangent to C at P is

$$y = \frac{1}{6}x + 1$$

(c) To show that the tangent and the circle do not intersect again, we need to show that the system of equations

$$3x^2 + 3y^2 - 2x + 6y = 9 \quad (1)$$

$$y = \frac{1}{6}x + 1 \quad (2)$$

only has one solution. Substituting (2) into (1), we have

$$3x^2 + 3\left(\frac{1}{6}x + 1\right)^2 - 2x + 6\left(\frac{1}{6}x + 1\right) = 9$$
$$\Rightarrow 108x^2 + 3(x+6)^2 - 72x + 36(x+6) = 324$$
$$\Rightarrow 108x^2 + 3x^2 + 36x + 108 - 72x + 36x + 216 - 324 = 0$$
$$\Rightarrow 111x^2 = 0$$

which only has one solution corresponding to $x = 0$. So the tangent only intersects the circle once.

Question 3 Solution.

We want the system of equations

$$(7-k)x + y = -1 \quad (1)$$

$$y = x^2 \quad (2)$$

to have no solutions.

$$\begin{aligned}&(7-k)x + x^2 = -1\\ &\Rightarrow x^2 + (7-k)x + 1 = 0\end{aligned}$$

For this to have no solutions, the discriminant needs to be negative, so

$$\begin{aligned}&(7-k)^2 - 4(1)(1) < 0\\ &\Rightarrow 49 - 14k + k^2 - 4 < 0\\ &\Rightarrow k^2 - 14k + 45 < 0\\ &\Rightarrow (k-9)(k-5) < 0\end{aligned}$$

and a quick sketch will show that is satisfied for $5 < k < 9$. The question asks for the *set* of integers k that solve the problem, and this is $\{k \in \mathbb{Z} : 5 < k < 9\}$. You can also express it as $\{k : k = 6, 7, 8\}$ for example.

Question 4 Solution.

(a) The equations of motion for the two particles are:

$$R_P(\uparrow^+): \quad T - 2g = 2a$$

and

$$R_Q(\downarrow^+): \quad 5g - T = 5a$$

Adding these two equations together gives

$$3g = 7a \Rightarrow a = \frac{3}{7}g$$

So the acceleration of the two particles is $\boxed{\frac{3}{7}g \text{ m s}^{-2}}$

(b) The particle Q is 3 m above the ground and moves to the ground from rest with constant acceleration, so this is a 'SUVAT' equation problem. Using $s = ut + \frac{1}{2}at^2$, we have

$$3 = 0(t) + \frac{1}{2}\left(\frac{3}{7}g\right)t^2 \Rightarrow t = \sqrt{\frac{42}{3g}}$$

and so it takes Q approximately $\boxed{\text{1.2 seconds}}$ to reach the ground.

(c) It is important to try and think about what the system does to break the problem down systematically. Once Q hits the ground, the string is no longer taut, so P will move under the influence of gravity until it reaches a maximum height and starts to fall down. To find the maximum height, we need to know the speed of P when Q hits the ground - this will be the same as the speed of Q when it hits the ground.

The speed of Q when it hits the ground is

$$v = \sqrt{2\left(\frac{3}{7}g\right)(3)} \approx 5.0199...$$

Therefore the maximum height P reaches above the ground can be solved using the 'SUVAT' equations with $s = ?$, $u = 5.0199...$, $v = 0$, $a = -9.8$, $t = \text{X}$.

$$0^2 = (5.0199...)^2 + 2(-9.8)(s) \Rightarrow s \approx 1.2857...$$

and so the maximum height P reaches above the ground is $3 + 3 + 1.285... = \boxed{7.3}$ m to 2 significant figures.

Good luck for the exam tomorrow!

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