CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 2**

Model Solution No: 1

(a) Using the quotient rule, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+3)(2) - 2x(2x)}{(x^2+3)^2}$$
$$= \frac{6-2x^2}{(x^2+3)^2}$$

The curve is increasing when it has positive derivative (or ≥ 0). Notice the denominator of our derivative is always positive, so the condition for the derivative to be positive is simply $6 - 2x^2 > 0$.

This inequality is equivalent to $3 - x^2 > 0$. The critical values are the values of x such that $3 - x^2 = 0 \Rightarrow x = \pm\sqrt{3}$. Then drawing a sketch will help to see the range of values of x that satisfy the inequality are $-\sqrt{3} < x < \sqrt{3}$. The final step is to translate this into set notation.

Answer: $\{x \in \mathbb{R} : -\sqrt{3} < x < \sqrt{3}\}$ OR $\{x \in \mathbb{R} : -\sqrt{3} \leq x \leq \sqrt{3}\}$ (or other equivalents)

(b) [Again, this is a challenging exercise.]

Solution: The first step is to differentiate the expression and find $\frac{\mathrm{d}y}{\mathrm{d}x}$. This is again best done using the quotient rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x^2)(\mathrm{e}^{-x^2})(-2x) - \mathrm{e}^{-x^2}(2x)}{(1+x^2)^2} \tag{1}$$

At this stage, there are two ways to proceed. Either you simplify (1) and then separately work out the right-hand side and show that it agrees with the left - this is easier. Alternatively, you show the result directly. We will do the latter because it is harder.

Now look at the result we want. We need to take out a factor of $-2x$ first, isolate a factor of y and then show the result inside the brackets. Let's continue from (1) and do this part first:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x\left(\frac{(1+x^2)(\mathrm{e}^{-x^2}) + \mathrm{e}^{-x^2}}{(1+x^2)^2}\right)\\ &= -2x\left(\underbrace{\frac{\mathrm{e}^{-x^2}}{(1+x^2)}}_{y} \times \frac{1+x^2+1}{1+x^2}\right)\\ &= -2xy\left(\frac{1+x^2}{1+x^2} + \frac{1}{1+x^2}\right) = -2xy\left(1+y\mathrm{e}^{x^2}\right)\end{aligned}$$

as required. Note in the final line, we have taken out the factor of y and decomposed the fraction $\dfrac{1+x^2+1}{1+x^2}$ by separating the numerator. Then we noticed that the first term is 1 and the second term is the same as $y\mathrm{e}^{x^2}$ (because the e^{x^2} cancels the numerator).

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Question Sheet: **Sheet 1**

Model Solution No: 2

(a) **Solution:** Suppose for a contradiction that $\frac{n}{s}$ is rational. Then we must have

$$\frac{n}{s} = \frac{p}{q}$$

for integers p, q and $q \neq 0$. However, this gives

$$s = \frac{nq}{p}$$

which tells us that s is rational (since $p \neq 0$ also). This contradicts the fact that s is irrational.

Hence $\frac{n}{s}$ must be irrational.

Extension: now try to prove that the sum of a rational number and irrational number is irrational. The process is identical.

(b) **Solution:** Suppose that there are a finite number of primes and let $\mathcal{P} = \{p_1, p_2, \cdots p_n\}$ be the collection of all the primes.

Consider the number $P = p_1 \times p_2 \times \cdots \times p_n + 1$. Neither p_1, p_2, $\cdots$, p_n divide P (since they leave a remainder of 1 upon division), but since P is a natural number, it must be divisible be some prime. Thus P itself must be prime, which contradicts the fact that $\mathcal{P}$ was the collection of all prime numbers. Hence there must be an infinite number of primes.

Alternative: A perhaps more elegant alternative is as follows:

Suppose that there are a finite number of primes. Then the largest prime number exists and call it p. Consider the number $P = p! + 1$. By construction, P is not divisible by any number less than or equal to p. Hence its only factors are P and 1 and thus P is prime. This contradicts the fact that p was the largest prime, and so we must have an infinite number of primes.

Question Sheet: **Sheet 1**

Model Solution No: 3

(a) Answer: $f(x) < \frac{4}{7}$ (allow y)

(b) Let $y = \frac{4x-1}{7x+2}$, then we have $y(7x+2) = 4x-1$ which re-arranges to $7xy-4x = -2y-1$. Factorising the LHS gives $x(7y-4) = -2y-1$ and thus $x = \frac{-2y-1}{7y-4}$.

Answer: $f^{-1}(x) = \frac{-2x-1}{7x-4}$ OR $f^{-1}(x) = \frac{2x+1}{4-7x}$ (or other equivalents)

(c) First work out $gf(x)$, which is

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g\left(\frac{4x-1}{7x+2}\right) \\ &= \ln\left(\frac{4x-1}{7x+2}+1\right) \end{aligned}$$

We want to solve $gf(x) = 5$, which (using the above) is equivalent to

$$\begin{aligned} &\ln\left(\frac{4x-1}{7x+2}+1\right) = 5 \\ &\Rightarrow \frac{4x-1}{7x+2}+1 = e^5 \\ &\Rightarrow \frac{4x-1}{7x+2} = e^5 - 1 \\ &\Rightarrow 4x-1 = (7x+2)(e^5-1) \\ &\Rightarrow 4x-1 = 7(e^5-1)x + 2(e^5-1) \quad \text{(by expanding the brackets)} \\ &\Rightarrow 4x - 7(e^5-1)x = 2e^5 - 1 \quad \text{(by taking all variable terms to one side)} \\ &\Rightarrow x(4-7e^5+7) = 2e^5-1 \quad \text{(by factorising)} \\ &\Rightarrow x = \frac{2e^5-1}{11-7e^5} \end{aligned}$$

Answer: $x = \dfrac{2e^5-1}{11-7e^5}$ OR $x = \dfrac{1-2e^5}{7e^5-11}$ (or other equivalents)

Note that however tempting it may be to use your calculator and use decimals, the question asks for an exact solution so you need to keep the exact values and grind through the algebra.

Question Sheet: **Sheet 1**

Model Solution No: 4

(a) The gradient of l_1 is $-\frac{1}{3}$. Since l_2 is perpendicular to l_1, its gradient is 3. We know it passes through the point $(4, 2)$, so its equation can be found using $y - y_1 = m(x - x_1)$:

$$y - 2 = 3(x - 4)$$

which gives $y = 3x - 10$.

Answer: $y = 3x - 10$

(b) For this, we need to find the points A, B, C and D.

The point A is where l_1 meets the y axis. Thus the point $A = (0, 5)$ (obtained by re-arranging for $y = mx + c$ or substituting in $x = 0$).

The point B is the point where l_1 and l_2 intersect. For the coordinates of this point, we solve their equations simultaneously. Thus the x coordinate of B satisfies

$$x + 3(3x - 10) = 15$$

which comes out as $x = \frac{9}{2}$. The y coordinate of B is then $\frac{7}{2}$. Thus $B = \left(\frac{9}{2}, \frac{7}{2}\right)$

The point C is where l_2 meets the x axis, which can be found by setting $y = 0$ in its equation. You obtain that $C = \left(\frac{10}{3}, 0\right)$.

The point D is where l_1 meets the x axis, which can be found by setting $y = 0$ in its equation. You obtain that $D = (15, 0)$.

Now we have all the information we need to find the two areas.

First we work out R_2. This is a triangle with base $CD = 15 - \frac{10}{3} = \frac{35}{3}$ and perpendicular height $\frac{7}{2}$. Thus the area of $R_2 = \frac{1}{2} \times \frac{35}{3} \times \frac{7}{2} = \frac{245}{12}$

Now the area of R_1 can be found by doing the area of triangle OAD subtract the area of triangle BCD (which is the region R_2). Now triangle OAD has area $\frac{1}{2} \times 5 \times 15 = \frac{75}{2}$. Thus the area of $R_1 = \frac{75}{2} - \frac{245}{12} = \frac{205}{12}$

Hence finally our ratio is

$$A(R_1) : A(R_2) = \frac{205}{12} : \frac{245}{12}$$

which simplifies to $41 : 49$.

Answer: $41 : 49$

Question Sheet: **Sheet 2**

Model Solution No: 5

(a) $6\cos\theta - 8\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$.

Thus comparing coefficients, we have $6 = R\cos\alpha$ and $8 = R\sin\alpha$.

Pythagoras gives $R^2 = 6^2 + 8^2 \Rightarrow R = 10$.

Dividing the two equations gives $\tan\alpha = \frac{4}{3} \Rightarrow \alpha = 0.9272... = 0.93$ (2 dp)

Answer: $6\cos\theta - 8\sin\theta = 10\cos(\theta + 0.93)$

(b) Linking the model to part (a), we have

$$H = 30 - 10\cos\left(\frac{\pi}{3}t + 0.93\right)$$

(i) the initial height of the passenger occurs at $t = 0$, which gives $H = 30 - 10\cos(0.93) = 24$ m (2 sf)

Answer: 24 m (above the ground). NB: we used the unrounded version of 0.93 in the actual calculation, which you should also do.

(ii) The maximum height of the passenger above the ground is $30 + 10 = 40$ m and the minimum height is $30 - 10 = 20$ m. Thus the diameter of the wheel is $40 - 20 = 20$ m and the radius is half this distance.

Answer: radius of wheel = 10 m

(c) You can do this in many ways. For example, you can solve the equation for a specific height and find the difference in time of the solutions corresponding to a whole cycle.

Alternatively: Cycles will occur every 2π radians because cos is 2π periodic. Hence we want

$$\frac{\pi}{3}t' + 0.93 = \frac{\pi}{3}t + 0.93 + 2\pi$$

where t' is another point in time. If this is satisfied, then the difference $t' - t$ corresponds to the length of a cycle. We can re-arrange:

$$\frac{\pi}{3}(t' - t) = 2\pi$$

and hence $t' - t = 6$. So the length of a cycle is 6 minutes.

Answer: 6 minutes

Question Sheet: **Sheet 2**

Model Solution No: 6

(a) Integrating wrt t gives

$$\mathbf{r} = (3t^2 - 4t + c)\mathbf{i} + \left(-\frac{5}{6}t^3 + d\right)\mathbf{j}$$

At time $t = 0$, we know that $r = 3\mathbf{i} - 2\mathbf{j}$, so

$$3\mathbf{i} - 2\mathbf{j} = c\mathbf{i} + d\mathbf{j}$$

Comparing components gives $c = 3$ and $d = -2$.

Thus at time t, we have that the displacement of P from O is

$$\mathbf{r} = (3t^2 - 4t + 3)\mathbf{i} + \left(-\frac{5}{6}t^3 - 2\right)\mathbf{j}$$

(b) When P is moving parallel to $\mathbf{j}$, the $\mathbf{i}$ component of its velocity must be 0 (draw a picture if you're not convinced!). Hence

$$6t - 4 = 0 \Rightarrow t = \frac{2}{3}$$

So its displacement when it is moving parallel to $\mathbf{j}$ is

$$\mathbf{r} = \left(3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 3\right)\mathbf{i} + \left(-\frac{5}{6}\left(\frac{2}{3}\right)^3 - 2\right)\mathbf{j} = \frac{5}{3}\mathbf{i} - \frac{182}{81}\mathbf{j}$$

and hence the distance from O is

$$\sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{182}{81}\right)^2}$$

which gives a distance of about 2.80 m.

Answer: 2.80 m

Question Sheet: **Sheet 2**

Model Solution No: 7

(a) Let X be the number of individuals that experience side effects. Then $X \sim \text{B}(100, 0.02)$

Answer: $X \sim \text{B}(100, 0.02)$ NB: $\text{B}(100, 0.02)$ is OK for the mark...

(b) **Answer:** Using your calculator, $\mathbb{P}(X > 5) = 0.0155$

(c) **Answer:** $\text{H}_0 : p = 0.02, \quad \text{H}_1 : p > 0.02$

(d) So we are approximating the binomial distribution $\text{B}(1000, 0.02)$ by a normal distribution. This will have a mean of $1000(0.02) = 20$ and variance of $1000(0.02)(0.98) = 19.6$. So now $X \sim N(20, 19.6)$.

Using our approximation and a continuity correction, our p-value is given by

$$\mathbb{P}(X \geq 27.5 | \text{H}_0) = 0.0451$$

(e) **Answer:** Since $0.0451 > 0.01$, the result is not significant at the 1 % level of significance. There isn't evidence to suggest that more than 2 % of drug users experience side effects.

(e) **Answer:** e.g. fluctuations due to sample size may impact the result / hypothesis testing is not certain/can be wrong / hypothesis testing does not explain the origin of the error

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