Worksheet 6 Solutions

Question 1 Solution.

(a) The form they want us to express our answer in indicates that we need to complete the square. Before we can do that, we need to get our expression in a suitable form:

$$\begin{aligned}4(2-3x)-6x^2+1 &= 8-12x-6x^2+1\\ &= -6(x^2+2x)+9\\ &= -6\left[(x+1)^2-1\right]+9\\ &= -6(x+1)^2+6+9\\ &= -6(x+1)^2+15\end{aligned}$$

which is in the form required and $a=-6$, $b=1$ and $c=15$. [NB: identification of a, b and c is not necessary but good practice.]

(b) The coordinates of the turning point are $\boxed{(-1,15)}$.

(c) This is a $\boxed{\text{maximum point}}$ (since the coefficient of x^2 is negative).

Question 2 Solution.

(a) Taking natural logs to both sides of $D = Ab^t$, we have:

$$\begin{aligned} &\ln D = \ln Ab^t \\ &\Rightarrow \ln D = \ln A + \ln b^t \qquad \text{(by product rule for logs)} \\ &\Rightarrow \ln D = \ln A + t\ln b \qquad \text{(by power rule for logs)} \\ &\Rightarrow \ln D = t\ln b + \ln A \end{aligned}$$

and so a graph of $\ln D$ against t is a straight line since the equation is of the form $\ln D = mt + c$, where m and c are constants.

(b) From our working above, we know that $\ln b$ is the gradient of the line l and $\ln A$ is the y-intercept. The gradient of l is given by

$$\begin{aligned} \frac{y_2 - y_1}{x_2 - x_1} &= \frac{2.20 - 0.40}{2 - 5} \\ &= -0.6 \end{aligned}$$

and so $\ln b = -0.6 \Rightarrow b = e^{-0.6} = 0.5488... \approx \boxed{0.55}$.

Now to find A, we need the y-intercept of l:

$$2.20 = -0.6(2) + c \Rightarrow c = 3.4$$

and so $\ln A = 3.4 \Rightarrow A = e^{3.4} = 29.964... \approx \boxed{30}$.

(c) The model is $D = (29.964...)e^{-0.6t}$. So the amount of drug in their system after 3 hours is $D = (29.964...)e^{-0.6(3)} = 4.953... \approx \boxed{5.0}$ mg.

(d) We want to solve

$$\begin{aligned} &\frac{D}{29.964...} = 0.05 = e^{-0.6t} \\ &\Rightarrow \ln 0.05 = \ln e^{-0.6t} \\ &\Rightarrow \ln 0.05 = -0.6t \\ &\Rightarrow t = -\frac{1}{0.6}\ln 0.05 = 4.9928... \end{aligned}$$

so the time taken for the amount of drug in the person's system to fall to 5% of the initial dose is approximately $\boxed{5.0}$ hours.

(e) At time $t = 0$ in this model, the amount of drug in the patient's system is

$$20 + (29.964...)e^{0.6(6)} = 10(2)^0 + C \Rightarrow C = 10.818...$$

so the value of C is approximately $\boxed{11}$ to 2 significant figures.

Question 3 Solution.

To find $f(x)$ from $f'(x)$, we need to integrate. But as usual, we need $f'(x)$ in index form to do this, so let's write $f'(x)$ in index form:

$$f'(x) = \frac{1-\sqrt{x}}{x^2} = \frac{1}{x^2} - \frac{x^{\frac{1}{2}}}{x^2} = x^{-2} - x^{-\frac{3}{2}}$$

. Hence,

$$\begin{aligned} f(x) &= \int f'(x)dx \\ &= \int \left(x^{-2} - x^{-\frac{3}{2}}\right) dx \\ &= -x^{-1} + 2x^{-\frac{1}{2}} + c \end{aligned}$$

Now since the curve passes through $(1, 8)$, we know that $f(1) = 8$, i.e.

$$8 = -1 + 2 + c \Rightarrow c = 7$$

and therefore

$$y = -x^{-1} + 2x^{-\frac{1}{2}} + 7$$

.

(b) First we need to find k:

$$k = -(4)^{-1} + 2(4)^{-\frac{1}{2}} + 7 = \frac{31}{4}$$

So now the distance between the points $A(1, 8)$ and $B(4, \frac{31}{4})$ is

$$|AB| = \sqrt{(1-4)^2 + \left(8 - \frac{31}{4}\right)^2} = \frac{\sqrt{145}}{4} \approx 3.01$$

Question 4 Solution.

(a)

$$\begin{aligned} v &= \int a dt \\ &= at + c \end{aligned}$$

Now at $t = 0$, $v = u$, so $u = a(0) + c \Rightarrow c = u$. So at any time t, we have that $v = u + at$ as required.

(b)

$$\begin{aligned} s &= \int v dt \\ &= \int (u + at) dt \\ &= ut + \frac{1}{2}at^2 + d \end{aligned}$$

Now at $t = 0$, $s = 0$, so $0 = u(0) + \frac{1}{2}a(0)^2 + d \Rightarrow d = 0$. So at any time t, we have that $s = ut + \frac{1}{2}at^2$ as required.

[**Common misconception.**: A lot of students think they can (or wonder why they can't just) integrate $\int v dt$ to $vt + c$ but you can't, since v is not a constant and is a function of t. On the other hand, a was a constant.]

(c) Our goal is to eliminate t from the equation. So we take $v = u + at$, write it as $t = \frac{v-u}{a}$ and wherever we see t in the second equation, we replace it with that:

$$\begin{aligned} s = ut + \frac{1}{2}at^2 &\Rightarrow s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2 \\ &\Rightarrow 2s = \frac{2uv - 2u^2}{a} + \frac{v^2 + u^2 - 2uv}{a} \\ &\Rightarrow 2as = 2uv - 2u^2 + v^2 + u^2 - 2uv \\ &\Rightarrow 2as = v^2 - u^2 \end{aligned}$$

as required.

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