CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 1**

Model Solution No: 1

(a) **Solution:** $f(-2) = 4(-2)^3 + 3(-2)^2 - 9(-2) + 2 = -32 + 12 + 18 + 2 = 0$. Therefore, since $f(-2) = 0$, we have that $(x+2)$ is a factor of $f(x)$.

(b) By inspection (or long division if you prefer), we have $f(x) = (x+2)(4x^2 - 5x + 1)$. This factorises further to $f(x) = (x+2)(4x-1)(x-1)$.

Answer: $f(x) = (4x-1)(x-1)(x+2)$

(c) Using part (a) and then doing some algebraic manipulation, we get:

$$\begin{aligned}
\frac{2x^2-3x+1}{4x^3+3x^2-9x+2} + \frac{2}{36x-9} &= \frac{(2x-1)(x-1)}{(4x-1)(x-1)(x+2)} + \frac{2}{9(4x-1)} \\
&= \frac{(2x-1)}{(4x-1)(x+2)} + \frac{2}{9(4x-1)} \\
&= \frac{9(2x-1)}{9(4x-1)(x+2)} + \frac{2(x+2)}{9(4x-1)(x+2)} \\
&= \frac{18x-9+2x+4}{9(4x-1)(x+2)} \\
&= \frac{20x-5}{9(4x-1)(x+2)} \\
&= \frac{5(4x-1)}{9(4x-1)(x+2)} \\
&= \frac{5}{9(x+2)}
\end{aligned}$$

Answer: $\dfrac{5}{9(x+2)}$ (the denominator expanded is fine too)

Question Sheet: **Sheet 1**

Model Solution No: 2

First we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, which we can get from the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

Note that $\frac{\mathrm{d}y}{\mathrm{d}t} = 5\mathrm{e}^t$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-6}{1-2t} = \frac{6}{2t-1}$ to give:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^t \times \left(\frac{2t-1}{6}\right)$$

We want the gradient at $x = 0$, but our gradient function is in terms of t, so we need to find out what t is at $x = 0$. That is simple: we just go to our x equation and see what happens when we demand $x = 0$:

$$0 = 3\ln(1-2t) \Rightarrow 1 = 1-2t \Rightarrow t = 0$$

Hence the gradient at $x = 0$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^0 \times \left(\frac{2(0)-1}{6}\right) = -\frac{5}{6}$$

Now at $x = 0$, $y = 5e^0 = 5$ (using again the fact that $t = 0$ at $x = 0$). Thus we have for the equation of the tangent:

$$y - 5 = -\frac{5}{6}(x-0)$$

or in the form required, $5x + 6y - 30 = 0$.

Answer: $5x + 6y - 30 = 0$ (or any non-zero integer multiple)

Question Sheet: **Sheet 1**

Model Solution No: 3

Solution: To integrate this, we first express the integrand in partial fractions. Thus we seek A and B such that

$$\frac{12}{(2t-3)(2t+3)} = \frac{A}{2t-3} + \frac{B}{2t+3}$$

Multiplying through by $(2t-3)(2t+3)$, we have

$$12 = A(2t+3) + B(2t-3)$$

Now this needs to be true for all t, so we can pick values of t (to our convenience) to help find A and B.

When $t = -\frac{3}{2}$, we have

$$12 = -6B \Rightarrow B = -2$$

Similarly when $t = \frac{3}{2}$, we have

$$12 = 6A \Rightarrow A = 2$$

Hence our integrand is

$$\frac{12}{4t^2-9} = \frac{2}{2t-3} - \frac{2}{2t+3}$$

Now let's work out the definite integral:

$$\begin{aligned}\int_2^{\frac{7}{2}} \frac{12}{4t^2-9}\,\mathrm{d}t &= \int_2^{\frac{7}{2}} \frac{2}{2t-3} - \frac{2}{2t+3}\,\mathrm{d}t \\ &= [\ln(2t-3) - \ln(2t+3)]_2^{\frac{7}{2}} \\ &= \ln(4) - \ln(10) - \ln(1) + \ln(7) = \ln\frac{(4)(7)}{10} = \ln\frac{14}{5}\end{aligned}$$

as required.

Remark 1: why did we choose those values of t to find A and B? As mentioned above, this must hold for all t, so you really can pick *any* two values of t. Try as an exercise to use two random values of t - you will get two simultaneous equations which you can solve fairly easily - your answer should be exactly the same. We picked the above values because they made our obtained equations easier (only in terms of one variable, so no need for simultaneous equations).

Remark 2: Why didn't you put mod signs around the log terms after integrating? Think about why we use mod terms usually - it is to ensure the argument is never non-positive. But in this case, they are redundant because the argument is always positive. You do need them in the most general case, but as said, they're redundant here.

Question Sheet: **Sheet 10**

Model Solution No: 4

(a) **Solution:** We prove this in the usual way - just we have 5 instead of the more general a:

$$S_N = 5 + 5r + 5r^2 + \cdots + 5r^{N-1} \quad (1)$$

Multiplying (1) by r, we get a second equation:

$$rS_N = 5r + 5r^2 + 5r^3 + \cdots + 5r^N \quad (2)$$

Notably, we can subtract (2) from (1) to get:

$$S_N - rS_N = 5 - 5r^N$$

because every other term cancels in the subtraction. By extract factors on both sides and then dividing, we then trivially have

$$S_N(1-r) = 5(1-r^N) \Rightarrow S_N = \frac{5(1-r^N)}{1-r}$$

which proves the claim as required.

(b) **Solution:** e.g. we require $|r| < 1$ because otherwise the series would not converge

(c) We know $S_\infty = 6$, i.e. $6 = \frac{5}{1-r}$. Solving this for r gives the result (which you can do for yourself).

Answer: $r = \frac{1}{6}$

Question Sheet: **Sheet 1**

Model Solution No: 5

(a) $\frac{\mathrm{d}h}{\mathrm{d}t} = -42$ when $h = 300$ and so

$$-42 = \frac{k}{10} \Rightarrow k = -420$$

Answer: $k = -420$

(b) Separating variables and then integrating both sides with respect to t, we have

$$\begin{aligned} &\sqrt{400-h}\ \frac{\mathrm{d}h}{\mathrm{d}t} = -420 \\ &\Rightarrow \int \sqrt{400-h}\,\mathrm{d}h = \int -420\,\mathrm{d}t \\ &\Rightarrow \frac{(400-h)^{\frac{3}{2}}}{\frac{3}{2}\times(-1)} = -420t + C \\ &\Rightarrow -\frac{2}{3}(400-h)^{\frac{3}{2}} = -420t + C \end{aligned}$$

Now at $t = 0$, $h = 400$ (the tank is completely full), so

$$0 = 0 + C \Rightarrow C = 0$$

Thus we have $-\frac{2}{3}(400-h)^{\frac{3}{2}} = -420t$.

You can re-arrange further, but the question doesn't ask for it, so it's unnecessary in this case. [SIDE NOTE: however, a lot of the time, the questions *do* ask you to re-arrange for the dependent variable, so read carefully.]

Answer: $-\frac{2}{3}(400-h)^{\frac{3}{2}} = -420t$ (or better)

(c) The tank is empty when $h = 0$, so

$$-420t = -\frac{2}{3}(400-0)^{\frac{3}{2}} \Rightarrow t = 12.698...$$

Answer: 12.7 seconds (3 sf)

Question Sheet: **Sheet 1**

Model Solution No: 6

(a) **Solution:** Using $v^2 = u^2 + 2as$ with $u = 5$, $a = 1.2$ and $s = 0.8$, we obtain $v^2 = 5^2 + 2(1.2)(0.8) = 26.92$

Therefore $v = \sqrt{26.92} = 5.188...$, which is 5.2 to 2 sf.

(b) This is a projectile motion question and these are best solved by splitting the motion into its components.

Note that the horizontal velocity of P is always 5.2 m/s (because no horizontal forces act to change this as it moves freely under gravity).

The vertical velocity of P at B is 0. It falls 1.5 m at 10 m/s^2, so we can find the vertical velocity at C using $v_y^2 = u^2 + 2as = 0^2 + 2(9.8)(1.5) \Rightarrow v_y = \frac{7}{5}\sqrt{15}$

Hence the speed of P at C is $\sqrt{v_x^2 + v_y^2} = \sqrt{5.188...^2 + \left(\frac{7}{5}\sqrt{15}\right)^2} = 7.5$ m/s (2 sf)

The direction is (by convention) described by the angle the particle's velocity makes with the horizontal. This is $\tan^{-1}\left(\frac{7\sqrt{15}/5}{5.188...}\right) = 46^{\circ}$ to 2 sf

Answer: At C, P moves with speed 7.5 m/s at an angle of 46° beneath the horizontal

(c) Time taken to move from B to C is

$$1.5 = \frac{1}{2}(10)t^2 \Rightarrow t = \sqrt{\frac{3}{10}}$$

We also need the time taken to move from A to B, which can be obtained using the SUVAT equations. For example, we will use $v = u + at$ to obtain $t = \frac{5.188...-5}{1.2} = 0.1570...$ (using the unrounded version of part (a)).

Thus the total time to move from A to C is $\sqrt{\frac{3}{10}} + 0.1570... = 0.70$ s.

Answer: 0.70 s.

(d) **Answer:** e.g. the acceleration of P on the table will now be less and so the speed of the particle at B will also be less, i.e. the answer to part (a) will be smaller. [Exercise: how will this affect part (b) and (c)?]

Question Sheet: **Sheet 1**

Model Solution No: 7

(a) **Answer:** some of the data is unavailable/non-numerical (so he must have cleaned these values to obtain numerical summary statistics)

(b) **Answer:** Mean = 4.87 hours of sunshine, standard deviation = 3.65 hours of sunshine (3 sf)

(c) **Answer:** e.g.

- The data must have either come from Lechaurs in 1987 or 2015 and these might not represent sunshine distributions in 2019
- data is available from May-October only so may not be a good representation outside of this period

(d) **Answer:** Use your calculator to find the probability is 0.486

(e) **Answer:** e.g. not a good assumption because the data is not symmetric

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