Worksheet 7 Solutions

Question 1 Solution.

(a) $5^x = 10 \Leftrightarrow x = \log_5(10) = 1.4306... \approx 1.43$ to 2 dp.

(b) This is a quadratic equation in disguise:

$$
\begin{aligned}
&2(4^x) - 5(2^x) + 3 = 0 \\
&\Rightarrow 2(2^{2x}) - 5(2^x) + 3 = 0 \\
&\Rightarrow (2(2^x) - 3)(2^x - 1) = 0 \\
&\Rightarrow 2(2^x) - 3 = 0 \text{ or } 2^x - 1 = 0 \\
&\Rightarrow 2^x = \frac{3}{2} \text{ or } 2^x = 1 \\
&\Rightarrow x = 0 \text{ or } x = \log_2\left(\frac{3}{2}\right) \approx 0.58
\end{aligned}
$$

So the solutions are approximately $\boxed{x = 0 \text{ or } x = 0.58}$.

[If you struggled to factorise the equation directly, try the substitution $y = 2^x$, but remember to reverse the substitution to find the values of x!]

Question 2 Solution.

(a) Using the binomial expansion, we have

$$\begin{aligned}\left(3-\frac{1}{\sqrt{x}}\right)^8 &= 3^8+\binom{8}{1}3^7\left(-\frac{1}{\sqrt{x}}\right)+\binom{8}{2}3^6\left(-\frac{1}{\sqrt{x}}\right)^2+\cdots \\ &= 6561-\frac{17495}{\sqrt{x}}+\frac{20412}{x}+\cdots\end{aligned}$$

(b)

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

so

$$\begin{aligned}\binom{n}{2} &= \frac{n!}{2!(n-2)!} \\ &= \frac{n\times(n-1)\times\cancel{(n-2)}\cdots\times\cancel{2}\times\cancel{1}}{2[\cancel{(n-2)}\times\cancel{(n-3)}\times\cdots\times\cancel{2}\times\cancel{1}]} \\ &= \frac{n(n-1)}{2}\end{aligned}$$

as required.

(c) So we know that

$$700=\binom{p}{2}(5)^2(1)^{p-2}$$

Using part (b) we obtain:

$$\begin{aligned}700 &= \frac{p(p-1)}{2}(25) \\ &\Rightarrow p^2-p-56=0 \\ &\Rightarrow (p-8)(p+7)=0\end{aligned}$$

and since $p>0$, we have that $p=8$

Question 3 Solution.

(a) We are given information about the maximum point on C - this indicates the use of differentiation. However, as usual, we first need to write $f(x)$ in index form:

$$f(x) = \frac{1}{\sqrt{x}} - \frac{p}{x^2} = x^{-\frac{1}{2}} - px^{-2}$$

Hence

$$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} + 2px^{-3}$$

Since $x = \sqrt[3]{32^2}$ is a maximum point, it satisfies $f'(\sqrt[3]{32^2}) = 0$,

$$\begin{aligned} f'(\sqrt[3]{32^2}) = 0 &\Rightarrow -\frac{1}{2}\left(\sqrt[3]{32^2}\right)^{-\frac{3}{2}} + 2p\left(\sqrt[3]{32^2}\right)^{-3} = 0 \\ &\Rightarrow -\frac{1}{2(32)} + \frac{2p}{(32^2)} = 0 \end{aligned}$$

So

$$p = \frac{32^2}{4(32)} = 8$$

as required. Alternatively, you could have substituted $p = 8$ into $f(x)$ and shown that the turning point occurs at $x = \sqrt[3]{32^2}$

(b) First we need to find where $f(x)$ crosses the x-axis:

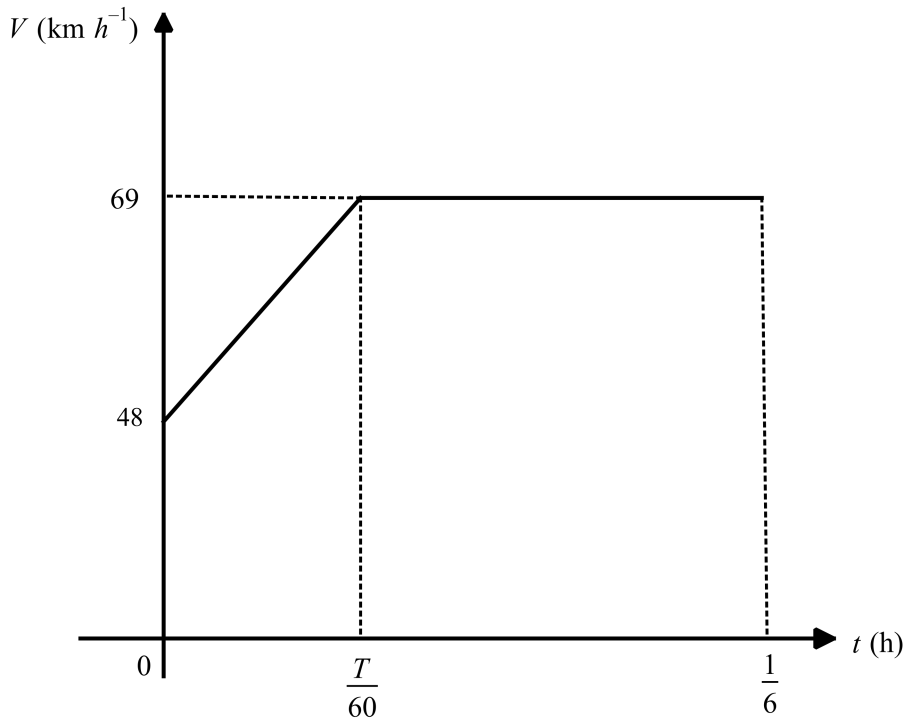
$$\frac{1}{\sqrt{x}} - \frac{8}{x^2} = 0 \Rightarrow x^{\frac{3}{2}} = 8 \Rightarrow x = 4$$

The area is given by

$$\begin{aligned} \text{Area} &= -\int_3^4 f(x)dx + \int_4^6 f(x)dx \\ &= -\int_3^4 (x^{-\frac{1}{2}} - 8x^{-2})dx + \int_4^6 (x^{-\frac{1}{2}} - 8x^{-2})dx \\ &= -\left[2\sqrt{x} + \frac{8}{x}\right]_3^4 + \left[2\sqrt{x} + \frac{8}{x}\right]_4^6 \\ &= -\left[2\sqrt{4} + \frac{8}{4} - 2\sqrt{3} - \frac{8}{3}\right] + \left[2\sqrt{6} + \frac{8}{6} - 2\sqrt{4} - \frac{8}{4}\right] \\ &= -\left(4 + 2 - 2\sqrt{3} - \frac{8}{3}\right) + \left(2\sqrt{6} + \frac{4}{3} - 4 - 2\right) \\ &= -8 + 2\sqrt{3} + 2\sqrt{6} \end{aligned}$$

Question 4 Solution.

(a) The speed-time graph looks like:



(b) The distance travelled is the area under the speed-time graph. The speed-time graph can be split into a trapezium and a rectangle, and so we can form the following equation:

$$\begin{aligned}
11.325 &= \text{Area under trapezium} + \text{Area under rectangle} \\
&= \frac{48+69}{2}\left(\frac{T}{60}\right) + 69\left(\frac{1}{6} - \frac{T}{60}\right) \\
&= \frac{39}{40}T + \frac{23}{2} - \frac{23}{20}T \\
&= \frac{23}{2} - \frac{7}{40}T
\end{aligned}$$

Then

$$T = \frac{40}{7}\left(\frac{23}{2} - 11.325\right) = 1$$

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