CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 6**

Model Solution No: 1

We start by using the product rule to find the first derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x\ln x + 4x^2\left(\frac{1}{x}\right) = 8x\ln x + 4x$$

Now the stationary point occurs when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Setting the first order derivative equal to zero, we have:

$$\begin{aligned} &8x\ln x + 4x = 0 \\ &\Rightarrow 4x(2\ln x + 1) = 0 \end{aligned}$$

Now $x \neq 0$ (since $x > 0$), so the stationary point occurs when $2\ln x + 1 = 0 \Rightarrow x = \mathrm{e}^{-\frac{1}{2}}$ (after some re-arranging).

We plug this back into the equation of the curve and find that the corresponding y coordinate at the stationary point is $y = 4(\mathrm{e}^{-\frac{1}{2}})^2\ln(\mathrm{e}^{-\frac{1}{2}}) = -2\mathrm{e}^{-1}$.

Thus the coordinates of stationary point is $(\mathrm{e}^{-\frac{1}{2}}, -2\mathrm{e}^{-1})$ (other equivalent forms are OK too, but they must be exact).

Now we need to determine the nature of the stationary point. As usual, we start by finding the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 8\ln x + 8x\left(\frac{1}{x}\right) + 4 = 8\ln x + 12$$

At $x = \mathrm{e}^{-\frac{1}{2}}$, we have

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\mathrm{e}^{-\frac{1}{2}}} = 8\ln(\mathrm{e}^{-\frac{1}{2}}) + 12 = -4 + 12 = 8$$

At our stationary point, the second derivative is positive and therefore the point is a minimum.

Answer: Stationary point has coordinates $(\mathrm{e}^{-\frac{1}{2}}, -2\mathrm{e}^{-1})$ and is a minimum.

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Model Solution No: 2

(a) Using the cosine rule, we have

$$BC^2 = 4^2 + 6^2 - 2(4)(6)\cos(12) = 5.0489...$$

and thus $BC = \sqrt{5.0489...} = 2.246...$ km

Answer: $BC = 2.0$ km (to the nearest 0.5 km)

(b) Now we can use the cosine rule again or we can use the sine rule. We will show both. Firstly, using the cosine rule, we have

$$\cos(ABC) = \frac{AB^2 + BC^2 - AC^2}{2(AB)(BC)} = \frac{4^2 + 2.246...^2 - 6^2}{2(4)(2.246...)} = -0.8317...$$

Hence angle $ABC = \cos^{-1}(-0.8317...) = 146.277....$

Then $\theta = 180 - 146.277... = 33.722...$

Answer: $\theta = 34^{\circ}$ (nearest degree)

Alternatively...: using the sine rule, we have

$$\frac{2.246...}{\sin 12} = \frac{6}{\sin ABC} \Rightarrow \sin ABC = 0.555...$$

Now remember, we are in the case of the ambiguous sine rule here, because we don't have the angle between the lines BC and AC (i.e. the ones we used in the formula). It is clear from the diagram that we need the obtuse angle and so

$$ABC = 180 - \sin^{-1}(0.55517...) = 146.277...$$

which if you take away from 180 gives you the required value in the same way as in the first method.

Question Sheet: **Sheet 6**

Model Solution No: 3

(a) Periodic with order 2 means that the sequence repeats itself every 2 terms. For example, the sequence $1, -1, 1, -1, \cdots$, is periodic with order 2, while the sequence $1, 2, 3, 1, 2, 3, \cdots$ is periodic with order 3.

Back to the problem at hand, we need to create an equation using the given information. Because the sequence is periodic with order 2 and since the first term is 1, we know that the third term must also be 1 (and the fifth, seventh, etc.).

Let's find an expression for the third term, u_3. Firstly $u_2 = ku_1 + 5 = k(1) + 5 = k + 5$.

Next $u_3 = ku_2 + 5 = k(k+5) + 5 = k^2 + 5k + 5$

But we know that $u_3 = 1$, thus

$$k^2 + 5k + 5 = 1 \Rightarrow k^2 + 5k + 4 = 0$$

which factorises nicely to $(k+4)(k+1) = 0$ and gives $k = -1$ or $k = -4$.

Answer: $k = -1$, $k = -4$.

(b) Usually with summation questions, the index is either small in which case you can just write out the terms or it is a larger number, where writing out and individually combining the terms would not be appropriate. In such cases, you need to use exploit properties of the sequence you're dealing with; here we exploit the fact that the sequence is periodic.

When $k = -1$, the sequence is $u_1 = 1, u_2 = 4, u_3 = 1, u_4 = 4, \cdots,$. Hence

$$\sum_{n=1}^{100} u_n = 1 + 4 + 1 + 4 + \cdots + 1 + 4$$

where the sum contains 100 terms.

Now you can think about this in a few ways. Either we have 50 lots of 1 and 50 lots of 4 to give a total of $50(1) + 50(4) = 250$ or you can notice that every two terms combine to give 5, so we have 50 lots of 5 which is 250 again. (They are of course equivalent ways of thinking about this problem.)

The case where $k = -4$ is even easier. We will leave you to figure out why the result is what we claim (hint: just write out the first few terms...).

Answer: when $k = -1$, the sum is 250. When $k = -4$, the sum is 100

More practice: Try working out these sums (hint: always a good idea to write out the first few terms and see the pattern):

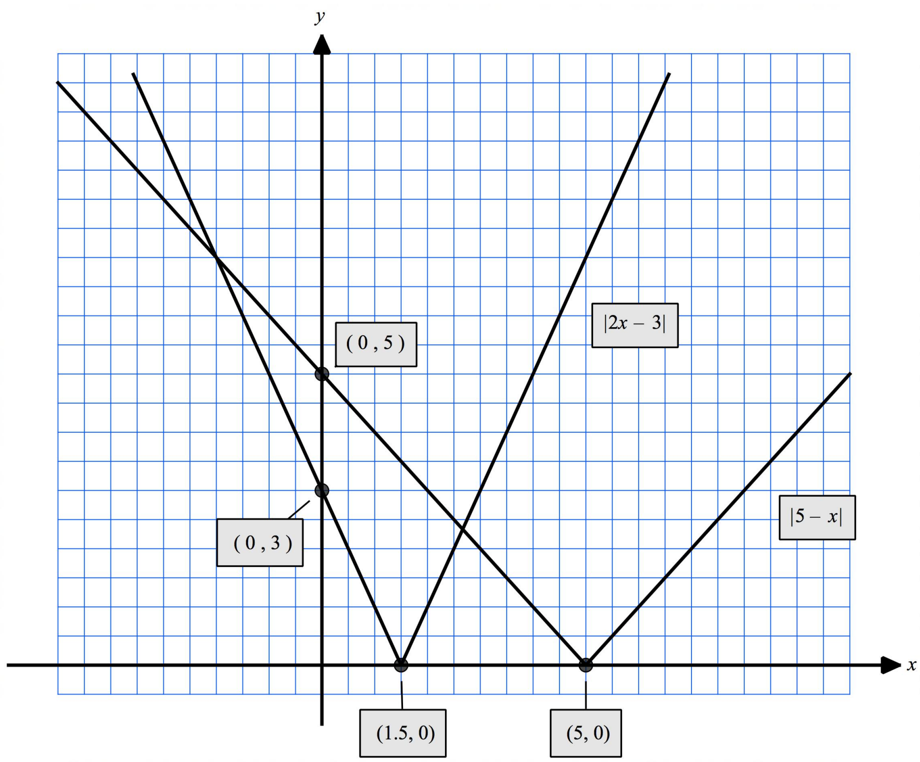
- (i) $\sum_{n=1}^{100} a_n$, where $a_{n+1} = \frac{4}{a_n}$ and $a_1 = 3$
- (ii) $\sum_{n=1}^{250} u_n$, where $u_n = 4 + (-1)^n$
- (iii) Given that $\sum_{n=1}^{50} x_n = 180$, what is $\sum_{n=1}^{50}(x_n + 2)$?

Answers: (i) $\frac{650}{3}$, (ii) 1000, (iii) 280

Question Sheet: **Sheet 6**

Model Solution No: 4

(a) (i) and (ii)



(b) The solutions to this equation correspond to the points of intersection of the two graphs.

The solution corresponding to the rightmost intersection point is given by

$$5 - x = 2x - 3$$

because those the LHS and RHS are the equations of the intersecting line segments. This has solution $x = \frac{8}{3}$

The solution corresponding to the leftmost intersection point is given by

$$5 - x = -(2x - 3)$$

because the LHS and RHS are the equations of the intersecting line segments. This has solution $x = -2$

Answer: $x = -2$ and $x = \frac{8}{3}$

Alternatives for (b): You can alternatively think of part (b) in a more algebraic fashion. If we have

$$|2x - 3| = |5 - x|$$

then by definition what is inside the moduli have to be equal up to sign. So

$$\pm(2x + 3) = \pm(5 - x)$$

You then have 4 cases to deal with, each of which has a corresponding solution. Two of these cases are not suitable though and once you have the solutions, you can plug them back in to see which ones work and which ones don't or using your graphs, immediately discount the unwanted equations.

The reason we get these unwanted solutions is in the same way we generate extra solutions when we square an equation: removing the moduli (or squaring both sides) expands the domain of our equation and thus the potential range of solutions.

OR: As an additional alternative, you can square both sides and solve the resulting equation. Note that squaring is not always a clean option and can get messy, so it is a good idea to understand the graphical and other approach of just using the definition as well.

Question Sheet: **Sheet 6**

Model Solution No: 5

(a) If the amount of chemical used forms an arithmetic progression, then the arithmetic progression must have first term 7 and common difference 1.2.

So after 25 experiments, the amount of chemical used will be

$$S_{25} = \frac{25}{2}[2(7) + (25-1)(1.2)] = 535 \text{ g}$$

Answer: 535 g

(b) If the amount of chemical used forms a geometric progression, then the geometric progression must have first term 7 and common ratio $\frac{8.2}{7} = \frac{41}{35}$.

So after 25 experiments, the amount of chemical used will be

$$S_{25} = \frac{7(1-(41/35)^{25})}{1-(41/35)} = 2091.77... \text{ g}$$

Answer: 2090 g (3 sf)

(c) **Model A:** In model A, we want to find the value of n such that

$$\begin{aligned} 1800 = \frac{n}{2}[2(7) + (n-1)(1.2)] &\Rightarrow 3600 = n(14 + 1.2n - 1.2) \\ &\Rightarrow 3600 = 12.8n + 1.2n^2 \\ &\Rightarrow 1.2n^2 + 12.8n - 3600 = 0 \end{aligned}$$

If you use the quadratic formula, you will find this has solutions $n = 49.69...$ and $n = -60.36...$. But since he can only perform a positive number of experiments, only the positive solution is valid.

Now for the golden question people seem to struggle with: is the final answer 49 or 50? Think about it: the chemical runs out on the (hypothetical) 49.69th experiment. Thus he cannot do the 50th one, so the maximum he can do is 49.

Model B: We want to find the value of n such that

$$1800 = \frac{7(1 - (41/35)^n)}{1 - (41/35)}$$

Re-arranging we find that

$$\left(\frac{41}{35}\right)^n = \frac{2209}{49}$$

Taking logs, we have $n \log \frac{41}{35} = \log \frac{2209}{49}$ and so $n = 24.07....$

Thus he only perform 24 experiments in the case of model B.

Answer: Model A gives a maximum of 49 experiments and model B gives a maximum of 24 experiments.

Question Sheet: **Sheet 6**

Model Solution No: 6

(a) Considering the crate and applying a force balance through N2L (Newton's 2nd Law) gives

$$20g - R = 20(1.8)$$

The LHS is because the resultant force acting on the crate in the direction of motion (downwards) is its weight minus the reaction. Note we don't need to worry about the tension in the string or resistance to motion because that is not affecting the crate directly. The RHS is mass times acceleration for the crate.

A common mistake is to think the acceleration of the crate is 0 because it is seemingly stationary on the floor. But this is not the case because the crate is moving together with the lift (if it's acceleration was 0, then it would start to float up as the lift descends...)

Solving the equation for R gives $R = 160$.

Answer: $R = 160$

(b) There are two ways to do this part. You can either consider the whole system or just the lift. We consider the whole system:

$$180g + 20g - 135 - T = 200(1.8)$$

which gives $T = 1465$ N for the tension in the string upon re-arranging.

Answer: 1465 N

Exercise: Obtain the same answer for the tension by considering just the lift. [Hint: there is a reaction force on the lift due to the crate that you have to consider.]

(c) **Answer:** e.g. the value for the acceleration due to gravity may be different in which case the lift and the crate will have a different weight

Question Sheet: **Sheet 6**

Model Solution No: 7

(a) **Answer:** e.g. it can be quicker / easier

(b) **Answer:** e.g. the 3 people may have refused to answer / the 3 people may not have taken their driving test

(c) Recall that for a histogram, area is proportional to frequency. There are 250 weighted (small) squares in the histogram that represent a total frequency of 50 people. Hence each small square represents $\frac{50}{250} = 0.2$ people. The number of squares between 25 and 38 is 93. Hence the total number of people that took between 25 and 38 hours to pass is $93 \times 0.2 = 18.6$.

Answer: either 18 or 19

(d) From left to right, each bar represents a frequency of 1, 4, 6, 5, 17, 14, 3 people. The median is the 25th value so lies in the class $30 - 40$. Now we can work out the median using interpolation:

$$Q_2 = 30 + \frac{9}{17}(10) = 35.294...$$

Answer: $Q_2 = 35.3$. If you used $(n + 1)/2$ for the median, that is acceptable too and you should have an answer of 35.6

NB: Other correct expressions for the median include

$$Q_2 = 40 - \frac{8}{17}(10)$$

or

$$\frac{40 - Q_2}{40 - 30} = \frac{33 - 25}{33 - 16}$$

(e) **Answer:** no because the data is not symmetric

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