CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 10**

Model Solution No: 1

(a) We have

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{2y}{y^2} = -\frac{2}{y}$$

And hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}y$$

Now we have to get this expression in terms of x. Using the given expression in this question, we have

$$\ln(y^2) = 1 - x$$
$$\Rightarrow y^2 = \mathrm{e}^{1-x}$$
$$\Rightarrow y = \sqrt{\mathrm{e}^{1-x}}$$

Hence we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\sqrt{\mathrm{e}^{1-x}}$$

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\sqrt{\mathrm{e}^{1-x}}$

(b) **Solution:** Using the quotient rule (oe), we have

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2\sin 2y}{\cos^2 2y}$$

And hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 2y}{2\sin 2y}y$$

Now we have to get this expression in terms of x. We can replace the $\cos^2 2y$ by $\frac{1}{x^2}$, but we need to do some work to replace the $\sin 2y$. Using the given expression in this question, we have

$$\cos 2y = \frac{1}{x}$$
$$\Rightarrow \cos^2 2y = \frac{1}{x^2}$$
$$\Rightarrow \sin^2 2y = 1 - \cos^2 2y = 1 - \frac{1}{x^2}$$
$$\Rightarrow \sin 2y = \sqrt{1 - \frac{1}{x^2}}$$

Hence we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{x^2}}{2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{2x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{2x\sqrt{x^2-1}}$$

Question Sheet: **Sheet 10**

Model Solution No: 2

(a) **Solution:** $a_2 = \dfrac{3-k}{2-k}$.

$$a_3 = \frac{3 - \frac{3-k}{2-k}}{2 - \frac{3-k}{2-k}} = \frac{6-3k-3+k}{4-2k-3+k} = \frac{3-2k}{1-k}$$

$$a_4 = \frac{3 - \frac{3-2k}{1-k}}{2 - \frac{3-2k}{1-k}} = \frac{3-3k-3+2k}{2-2k-3+2k} = \frac{-k}{-1} = k \text{ and thus } a_1 = a_4.$$

(b) This sequence is periodic with order 3. The terms within the sequence are $5, \frac{2}{3}, \frac{7}{4}$. The 5 occurs at a_1, a_4, a_7, a_{10} etc.

Hence

$$\sum_{r=9}^{100} a_r = \underbrace{\frac{7}{4}}_{a_9} + 30\underbrace{\left(5 + \frac{2}{3} + \frac{7}{4}\right)}_{a_{10}+a_{11}+\cdots+a_{99}} + \underbrace{5}_{a_{100}} = \frac{917}{4}$$

Answer: $\frac{917}{4}$

Question Sheet: **Sheet 10**

Model Solution No: 3

(a) (i) **Answer:** $5\sqrt{5}$

(ii) **Solution:** The diameter of C_2 is $5\sqrt{5} - \sqrt{5} - \sqrt{20} = 2\sqrt{5}$. Hence the radius of C_2 is $\sqrt{5}$

(b) The line that passes through PR also passes through Q. Now this has gradient $\frac{0 - -5}{6 - -4} = \frac{5}{10} = \frac{1}{2}$

Since the line passes through $(6, 0)$, its equation is given by

$$y - 0 = \frac{1}{2}(x - 6) \Rightarrow 2y = x - 6$$

Then Q lies on the y axis, so it's y coordinate is given by $2y = 0 - 6 \Rightarrow y = -3$.

So coordinates of Q are $(0, -3)$.

Answer: $(0, -3)$.

(c) **Answer:** $x^2 + (y + 3)^2 = 5$

Question Sheet: **Sheet 10**

Model Solution No: 4

(a) **Answer:** e.g. f is not one-to-one

(b) **Solution:** Using the product rule,

$$f'(x) = 10(-2)\mathrm{e}^{-2x}\sin^2(x) + 10(2)\mathrm{e}^{-2x}(\sin(x))(\cos x)$$

Taking out a factor of $20\mathrm{e}^{-2x}\sin(x)$, we get

$$f'(x) = 20\mathrm{e}^{-2x}\sin(x)(-\sin(x) + \cos(x))$$

Now the x coordinate of P satisfies $f'(x) = 0$. Since $\mathrm{e}^{-2x} \neq 0$ and $\sin(x) = 0$ implies $x = 0$ or $x = \pi$ (which is clearly not the case - see diagram and domain of function), we must have

$$\cos(x) - \sin(x) = 0$$

as required

(c) We have $\cos x - \sin x = 0$ which gives $\tan x = 1$ and thus, for our domain, the only possibility is $x = \frac{\pi}{4}$.

Putting this back into the equation of our curve, we have $y = 5\mathrm{e}^{-\frac{\pi}{2}}$

(d) (i) **Answer:** $0 \leq y \leq 15\mathrm{e}^{-\frac{\pi}{2}}$

(d) (ii) **Answer:** $0 \leq y \leq 5\mathrm{e}^{-\frac{\pi}{2}}$

Question Sheet: **Sheet 10**

Model Solution No: 5

(a) **Answer:** 1.04

(b) **Answer:** $30000 \times 1.04^2 = 32448$

(c) Want to know when

$$\begin{aligned} &30000 \times (1.04)^{N-1} > 50000 \\ &\Rightarrow 1.04^{N-1} > \frac{5}{3} \\ &\Rightarrow (N-1)\log 1.04 > \log\frac{5}{3} \\ &\Rightarrow N > \frac{\log(5/3)}{\log 1.04} + 1 = 14.024... \end{aligned}$$

Hence the year the population first exceeds 50000 is Year 15.

Answer: Year 15

(d) (i) **Answer:** Model predicts $n = 1$ as 30000, $n = 2$ as 31200, $n = 3$ as 32448, $n = 4$ as 33746, $n = 5$ as 35096. Thus the model is not a good for this town because it overestimates the population for $n = 2$, $n = 3$, $n = 4$ and $n = 5$.

(ii) **Answer:** e.g. the data suggests a saturation/upper limit of the results, while the model does not saturate/tend to an upper limit

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