FP1 – Proof by mathematical induction Questions

8. (a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5)$$

(6)

(b) A sequence of positive integers is defined by

$$u_1 = 1,$$
$$u_{n+1} = u_n + n(3n+1), \qquad n \in \mathbb{Z}^+$$

Prove by induction that

$$u_n = n^2(n-1) + 1, \qquad n \in \mathbb{Z}^+$$

(5)

10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 2^{2n-1} + 3^{2n-1} \text{ is divisible by 5.}$$

(6)

6. (a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8)$$

(3)

(c) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$.

(3)

7. A sequence can be described by the recurrence formula

$$u_{n+1} = 2u_n + 1, \qquad n \geqslant 1, \quad u_1 = 1$$

(a) Find u_2 and u_3.

(2)

(b) Prove by induction that $u_n = 2^n - 1$

(5)

9. Prove by induction, that for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$,

(6)

(b) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)

9. A sequence of numbers $u_1, u_2, u_3, u_4, \ldots$ is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}\left(4^n - 1\right)$$

(5)

7. $$f(n) = 2^n + 6^n$$

(a) Show that $f(k+1) = 6f(k) - 4(2^k)$.

(3)

(b) Hence, or otherwise, prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 8.

(4)

9. (a) Prove by induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$,

(b) show that

$$\sum_{r=1}^{n} (r+2)(r+3) = \frac{1}{3}n(n^2+an+b),$$

where a and b are integers to be found.

(5)

(c) Hence show that

$$\sum_{r=n+1}^{2n} (r+2)(r+3) = \frac{1}{3}n(7n^2+27n+26)$$

(3)