KS5 "Full Coverage": Proof By Induction

The Full Coverage worksheets are designed to cover one question of each type seen in past papers, for each A Level topic.

Question 1

Categorisation: Simple divisibility proofs.

[Edexcel FP1 June 2011 Q9b Edited]

Prove by induction, that for $n \in \mathbb{Z}^+$,

$f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6 marks)

Question 2

Categorisation: Divisibility proofs with two powers.

[Edexcel FP1 June 2014 Q9 Edited]

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 8^n - 2^n$$

is divisible by 6

(6 marks)

Question 3

Categorisation: Divisibility proofs involving more complex powers.

[Edexcel FP1(Old) June 2017 Q9ii]

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 3^{3n-2} + 2^{3n+1} \text{ is divisible by } 19$$

(6)

Question 4

Categorisation: Summation proofs.

[Edexcel FP1(Old) June 2015 Q6ii]

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n\left(4n^2-1\right)$$

Question 5

Categorisation: Summation proofs involving algebraic fractions.

[Edexcel FP1(Old) June 2016 Q8i]

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

(5)

Question 6

Categorisation: Recurrence relations where there is one previous term.

[Edexcel FP1(Old) June 2016 Q8ii]

A sequence of positive rational numbers is defined by

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3}u_n + \frac{8}{9}, \qquad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3}\right)^n + \frac{4}{3}$$

(5)

Question 7

Categorisation: Recurrence relation proofs based on both the previous term and *n*.

[Edexcel FP1 June 2013 Q9a Edited]

A sequence of numbers is defined by

$u_1 = 8$
$u_{n+1} = 4u_n - 9n, \ n \geq 1$

Prove by induction that, for $n \in \mathbb{Z}^+, \ \ u_n = 4^n + 3n + 1$

(5 marks)

Question 8

Categorisation: Recurrence relations where there are two previous terms.

[Edexcel FP1(Old) June 2017 Q9i]

A sequence of numbers is defined by

$$u_1 = 6, \qquad u_2 = 27$$

$$u_{n+2} = 6u_{n+1} - 9u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 3^n(n+1)$$

(6)

Question 9

Categorisation: Matrix proofs.

[Edexcel FP1(Old) June 2016 Q6i]

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

Question 10

Categorisation: Proof by induction using integration by parts

[Edexcel P6 June 2004 Q4]

(a) Prove by induction that

$$\frac{d^n}{dx^n}(e^x \cos x) = 2^{\frac{1}{2}n} e^x \cos\left(x + \tfrac{1}{4}n\pi\right), \quad n \geq 1.$$

(8)

(b) Hence find the Maclaurin series expansion of $e^x \cos x$, in ascending powers of x, up to and including the term in x^4.

(3)

Question 11

Categorisation: Inequalities.

Prove $4^{n-1} > n^2$ for $n \geq 3$ by mathematical induction. *[Source: iitutor]*

Answers

Question 1

$f(1) = 7^{2-1} + 5 = 7 + 5 = 12$,	Shows that $f(1) = 12$.	B1
{which is divisible by 12}. $\{\therefore f(n)$ is divisible by 12 when $n = 1.\}$		
Assume that for $n = k$, $f(k) = 7^{2k-1} + 5$ is divisible by 12 for $k \in ¢^{+}$.		
So, $f(k+1) = 7^{2(k+1)-1} + 5$	Correct unsimplified expression for $f(k+1)$.	B1
giving, $f(k+1) = 7^{2k+1} + 5$		
$\therefore f(k+1) - f(k) = \left(7^{2k+1} + 5\right) - \left(7^{2k-1} + 5\right)$	Applies $f(k+1) - f(k)$. No simplification is necessary and condone missing brackets.	M1
$= 7^{2k+1} - 7^{2k-1}$		
$= 7^{2k-1}\left(7^2 - 1\right)$	Attempting to isolate 7^{2k-1}	M1
$= 48\left(7^{2k-1}\right)$	$48\left(7^{2k-1}\right)$	A1cso
$\therefore f(k+1) = f(k) + 48\left(7^{2k-1}\right)$, which is divisible by 12 as both $f(k)$ and $48\left(7^{2k-1}\right)$ are both divisible by 12.(1) If the result is true for $n = k$, (2) then it is now true for $n = k+1$. (3) As the result has shown to be true for $n = 1$,(4) then the result is true for all n. (5). **All 5 aspects need to be mentioned at some point for the last A1.**	Correct conclusion with no incorrect work. Don't condone missing brackets.	A1 cso

Question 2

$f(1) = 8^1 - 2^1 = 6,$	Shows that $f(1) = 6$	B1
Assume that for $n = k$, $f(k) = 8^k - 2^k$ is divisible by 6.		
$f(k+1) - f(k) = 8^{k+1} - 2^{k+1} - \left(8^k - 2^k\right)$	Attempt $f(k+1) - f(k)$	M1
$= 8^k(8-1) + 2^k(1-2) = 7 \times 8^k - 2^k$		
$= 6 \times 8^k + 8^k - 2^k \left(= 6 \times 8^k + f(k)\right)$	M1: Attempt $f(k+1) - f(k)$ as a multiple of 6 A1: rhs a correct multiple of 6	M1A1
$f(k+1) = 6 \times 8^k + 2f(k)$	Completes to $f(k+1) =$ a multiple of 6	A1
If the result is **true for** $n = k$, then it is now **true for** $n = k+1$. As the result has been shown to be **true for** $n = 1$, then the result is **true for all** n $(\in \square^{+}.)$		A1cso
	Do not award final A if n defined incorrctly e.g. 'n is an integer' award A0	

[6]

Question 3

$f(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}. {$\therefore f(n)$ is divisible by 19 when $n = 1$ }	Shows $f(1) = 19$	B1
Assume that for $n = k$, $f(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$.		
$f(k+1) = 3^{3(k+1)-2} + 2^{3(k+1)+1}$	Applies $f(k+1)$ with at least 1 power correct	M1
$f(k+1) = 27(3^{3k-2}) + 8(2^{3k+1})$ $= 8(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ **or** $= 27(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$	Either $8(3^{3k-2} + 2^{3k+1})$ or $8f(k)$; $19(3^{3k-2})$ or $27(3^{3k-2} + 2^{3k+1})$ or $27f(k)$; $-19(2^{3k+1})$	A1; A1
$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ **or** $f(k+1) = 27f(k) - 19(2^{3k+1})$	**Dependent on at least one of the previous accuracy marks being awarded.**	dM1
{$\therefore f(k+1) = 8f(k) + 19(3^{3k-2})$ is divisible by 19 as both $8f(k)$ and $19(3^{3k-2})$ are both divisible by 19}		
If the result is **true for $n = k$**, then it is now **true for $n = k + 1$**. As the result has shown to be **true for $n = 1$**,then the result is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for n = 1 stated earlier.	A1 cso

[6]

Question 4

If $n=1$, $\sum_{r=1}^{n}(2r-1)^2=1$ and $\frac{1}{3}n(4n^2-1)=1$, so **true** for $\boldsymbol{n}$ **= 1**.	B1
Assume result true for $n=k$ so $\sum_{r=1}^{k+1}(2r-1)^2=\frac{1}{3}k(4k^2-1)+(2(k+1)-1)^2$	M1
$=\sum_{r=1}^{k+1}(2r-1)^2=\frac{1}{3}(2k+1)\{(2k^2-k)+(3(2k+1))\}$	M1 A1
$=\frac{1}{3}(2k+1)\{(2k^2+5k+3)\}$ or $\frac{1}{3}(k+1)(4k^2+8k+3)$ or $\frac{1}{3}((2k+3)(2k^2+3k+1)\}$	
$=\frac{1}{3}(k+1)(2k+1)(2k+3)$ $\quad=\frac{1}{3}(k+1)(4(k+1)^2-1)$	dA1
True for $\boldsymbol{n=k+1}$ if **true** for $\boldsymbol{n=k}$, (and **true** for $\boldsymbol{n}$ **= 1**) so **true** by induction for all $\boldsymbol{n}\in\boldsymbol{Z}^+$	A1cso
	(6)

Question 5

If $n=1$, $\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=\frac{3}{4}$ **and** $1-\frac{1}{(n+1)^2}=\frac{3}{4}$, so true for $n=1$.	B1
Assume result true for $n=k$ **and** consider $\sum_{r=1}^{k+1}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(k+1)^2}+\frac{2(k+1)+1}{(k+1)^2(k+2)^2}$	M1
$=1-\left(\frac{(k+2)^2}{(k+1)^2(k+2)^2}-\frac{2(k+1)+1}{(k+1)^2(k+2)^2}\right)=1-\left(\frac{(k^2+2k+1)}{(k+1)^2(k+2)^2}\right)$	A1
$=1-\left(\frac{(k+1)^2}{(k+1)^2(k+2)^2}\right)=1-\left(\frac{1}{(k+1+1)^2}\right)$	M1
True for $\boldsymbol{n=k+1}$ if **true** for $\boldsymbol{n=k}$, (and **true** for $\boldsymbol{n}$ **= 1**) so **true** by induction for all $\boldsymbol{n}\in\boldsymbol{Z}^+$	A1cso
	(5)

Question 6

$n=1$: $u_1=5\times\left(\frac{1}{3}\right)^1+\frac{4}{3}=3$ so expression for u_n true for $n=1$

Assume result true for $n=k$ **and** consider $u_{k+1}=\frac{1}{3}(5\times\left(\frac{1}{3}\right)^k+\frac{4}{3})+\frac{8}{9}$

Obtain $u_{k+1}=5\times\left(\frac{1}{3}\right)^{k+1}+\frac{4}{9}+\frac{8}{9}$

$5\times\left(\frac{1}{3}\right)^{k+1}+\frac{4}{3}$ and deduce that result is true for $n=k+1$

True for $\boldsymbol{n=k+1}$ if **true** for $\boldsymbol{n=k}$, (and **true** for $\boldsymbol{n}$ **= 1**) so **true** by induction for all $\boldsymbol{n}\in\boldsymbol{Z}^+$

Question 7

$u_1 = 8$ given $n = 1 \Rightarrow u_1 = 4^1 + 3(1) + 1 = 8$ ($\therefore$ true for $n = 1$)	$4^1 + 3(1) + 1 = 8$ seen	B1
Assume true for $n = k$ so that $u_k = 4^k + 3k + 1$		
$u_{k+1} = 4\left(4^k + 3k + 1\right) - 9k$	Substitute u_k into u_{k+1} as $u_{k+1} = 4u_k - 9k$	M1
$= 4^{k+1} + 12k + 4 - 9k = 4^{k+1} + 3k + 4$	Expression of the form $4^{k+1} + ak + b$	A1
$= 4^{k+1} + 3\left(k + 1\right) + 1$	Correct completion to an expression in terms of $k + 1$	A1
If <u>true for $n = k$</u> then <u>true for $n = k + 1$</u> and as <u>true for $n = 1$</u> <u>true for all n</u>	Conclusion with all 4 underlined elements that can be seen anywhere in the solution; n defined incorrectly award A0.	A1 cso

Question 8

$u_{n+2} = 6u_{n+1} - 9u_n$, $n \geqslant 1$, $u_1 = 6$, $u_2 = 27$; $u_n = 3^n(n+1)$		
$n = 1$; $u_1 = 3(2) = 6$	Check that $u_1 = 6$ and $u_2 = 27$	B1
$n = 2$; $u_2 = 3^2(2+1) = 27$		
So u_n is true when $n = 1$ and $n = 2$.		
Assume that $u_k = 3^k(k+1)$ and $u_{k+1} = 3^{k+1}(k+2)$ are true.	Could assume for $n = k, n = k-1$ and show for $n = k+1$	
Then $u_{k+2} = 6u_{k+1} - 9u_k$		
$= 6(3^{k+1})(k+2) - 9(3^k)(k+1)$	Substituting u_k and u_{k+1} into $u_{k+2} = 6u_{k+1} - 9u_k$	M1
	Correct expression	A1
$= 2(3^{k+2})(k+2) - (3^{k+2})(k+1)$	Achieves an expression in 3^{k+2}	M1
$= (3^{k+2})(2k + 4 - k - 1)$		
$= (3^{k+2})(k+3)$		
$= (3^{k+2})(k+2+1)$	$(3^{k+2})(k+2+1)$ or $(3^{k+2})(k+3)$	A1
If the result is **true for $n = k$ and $n = k+1$** then it is now **true for $n = k+2$**. As it is **true for $n = 1$ and $n = 2$** then it is **true for all n** $(\in \mathbb{Z}^+)$.	Correct conclusion seen at the end. Condone true for $n = 1$ and $n = 2$ seen anywhere. This should be compatible with assumptions.	A1 cso
		[6]

Question 9

If $n=1$, $\begin{pmatrix}1 & 0\\ -1 & 5\end{pmatrix}^1 = \begin{pmatrix}1 & 0\\ -\frac{1}{4}(5^1-1) & 5^1\end{pmatrix}$ so **true** for $\boldsymbol{n=1}$	B1	
Assume result true for $n=k$ $\begin{pmatrix}1 & 0\\ -1 & 5\end{pmatrix}^{k+1} = \begin{pmatrix}1 & 0\\ -\frac{1}{4}(5^k-1) & 5^k\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 5\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ -1 & 5\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{4}(5^k-1) & 5^k\end{pmatrix}$	M1	
$\begin{pmatrix}1 & 0\\ -1 & 5\end{pmatrix}^{k+1} = \begin{pmatrix}1 & 0\\ -\frac{1}{4}(5^k-1)-5^k & 5\times5^k\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ -1-5.\frac{1}{4}(5^k-1) & 5\times5^k\end{pmatrix}$	M1 A1	
$=\begin{pmatrix}1 & 0\\ -\frac{1}{4}5^k+\frac{1}{4}-5^k & 5^{k+1}\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ -1-\frac{1}{4}5^{k+1}+\frac{5}{4} & 5^{k+1}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1}\end{pmatrix}$	A1	
True for $\boldsymbol{n=k+1}$ if **true** for $\boldsymbol{n=k}$, (and **true** for $\boldsymbol{n=1}$) so **true** by induction for all $\boldsymbol{n\in Z^+}$.	A1cso	(6)

Question 10

(a)	$n=1$: $\frac{d}{dx}(e^x\cos x) = e^x\cos x - e^x\sin x$ (Use of product rule)	M1	
	$\cos\left(x+\frac{\pi}{4}\right) = \cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(\cos x - \sin x)$	M1	
	$\frac{d}{dx}(e^x\cos x) = 2^{\frac{1}{2}}e^x\cos\left(x+\frac{\pi}{4}\right)$ True for $n=1$ (c.s.o. + comment)	A1	
	Suppose true for $n=k$.		
	$\left[\frac{d^{k+1}}{dx^{k+1}}(e^x\cos x)\right] = \frac{d}{dx}\left(2^{\frac{1}{2}k}e^x\cos\left(x+\frac{k\pi}{4}\right)\right)$	M1	
	$=2^{\frac{1}{2}k}\left[e^x\cos\left(x+\frac{k\pi}{4}\right) - e^x\sin\left(x+\frac{k\pi}{4}\right)\right]$	A1	
	$=2^{\frac{1}{2}k}e^x\sqrt{2}\cos\left(x+\frac{k\pi}{4}+\frac{\pi}{4}\right) = 2^{\frac{1}{2}(k+1)}e^x\cos\left(x+(k+1)\frac{\pi}{4}\right)$	M1 A1	
	$\therefore$ True for $n=k+1$, so true (by induction) for all n. (≥ 1)	A1(cso)	(8)
(b)	$1+\left(\sqrt{2}\cos\frac{\pi}{4}\right)x + \frac{1}{2}\left(2\cos\frac{\pi}{2}\right)x^2 + \frac{1}{6}\left(2\sqrt{2}\cos\frac{3\pi}{4}\right)x^3 + \frac{1}{24}(4\cos\pi)x^4$	M1	
	(1) (0) (−2) (−4)		
	$e^x\cos x = 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4$ (or equiv. fractions)	A2(1,0)	(3)
			11

Question 11

Step 1: Show it is true for $n = 3$.

LHS $= 4^{3-1} = 16$

RHS $= 3^2 = 9$

LHS > RHS

Therefore it is true for $n = 3$.

Step 2: Assume that it is true for $n = k$.

That is, $4^{k-1} > k^2$.

Step 3: Show it is true for $n = k + 1$.

That is, $4^k > (k+1)^2$.

$$
\begin{aligned}
\text{LHS} &= 4^k \\
&= 4^{k-1+1} \\
&= 4^{k-1} \times 4 \\
&> k^2 \times 4 && \text{by the assumption } 4^{k-1} > k^2 \\
&= k^2 + 2k^2 + k^2 && 2k^2 > 2k \text{ and } k^2 > 1 \text{ for } k \geq 3 \\
&> k^2 + 2k + 1 \\
&= (k+1)^2 \\
&= \text{RHS}
\end{aligned}
$$

$\text{LHS} > \text{RHS}$

Therefore it is true for $n = k + 1$ assuming that it is true for $n = k$.

Therefore $4^{n-1} > n^2$ is true for $n \geq 3$.