CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 9**

Model Solution No: 1

Solution: Since $(x-2)$ is a factor, we can find a using the factor theorem. The factor theorem tells us that $f(2)=0$ and thus

$$\begin{aligned}
&2(2)^3+a(2)^2+18(2)-24=0\\
&\Rightarrow 16+4a+36-24=0\\
&\Rightarrow 4a=-28\\
&\Rightarrow a=-7
\end{aligned}$$

So we have $f(x)=2x^3-7x^2+18x-24$. By inspection (or long division if you prefer), we then have $f(x)=(x-2)(2x^2-3x+12)$.

To show that $f(x)=0$ only has one real solution, it remains to show that the equation $2x^2-3x+12=0$ has no real solutions (because we already have one real solution which is $x=2$ and so the quadratic factor must have none).

The discriminant of $2x^2-3x+12$ is $(-3)^2-4(2)(12)=-87<0$. Since the discriminant is less than 0, the equation $2x^2-3x+12=0$ admits no real roots.

Hence $f(x)=0$ only has one real root (which is $x=2$).

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Question Sheet: **Sheet 9**

Model Solution No: 2

(a) **Solution:** For example take $a = 1$ and $b = -1$, then $|2a + 3b| = |2(1) + 3(-1)| = |-1| = 1$. However, $2|a| + 3|b| = 2|1| + 3|-1| = 5$. Therefore, it is not necessarily true that $|2a + 3b| = 2|a| + 3|b|$.

Any value of a and b where one is positive and one is negative works.

Extension: can you see why? Even better, can you prove it? Hint: it is easier to show that the statement is true in the other cases...

(b) **Solution:** We prove this by exhaustion. We have the cases $n = 1$, $n = 2$, $n = 3$, $n = 4$ and $n = 5$. It is easiest/clearer to do this using a table (but this is not a requirement):

n	3^n	Inequality direction	$1 + 2n$
1	3	$\geq$	2
2	9	$\geq$	5
3	27	$\geq$	7
4	81	$\geq$	9
5	243	$\geq$	11

Hence in each case, we have $3^n \geq 1 + 2n$ so by exhaustion, we have $3^n \geq 1 + 2n$ for all $n \in \mathbb{N}, n \leq 5$.

(c) **Solution:** Suppose that $\sqrt{7}$ is rational. Then we can write

$$\sqrt{7} = \frac{a}{b}$$

where $a, b \in \mathbb{Z}$, $b \neq 0$ and $\gcd(a, b) = 1$.

Then we have

$$7 = \frac{a^2}{b^2} \Rightarrow a^2 = 7b^2$$

Thus a^2 is a multiple of 7. This implies that a is a multiple of 7, so let $a = 7k$ for some $k \in \mathbb{Z}$. Then

$$(7k)^2 = 7b^2 \Rightarrow b^2 = 7k^2$$

Thus b^2 is a multiple of 7 as well and so b is a multiple of 7. But this contradicts that $\gcd(a, b) = 1$. Hence $\sqrt{7}$ is irrational.

(d) **Solution:** Suppose that $m + n\sqrt{7}$ is rational. Then we can write

$$m + n\sqrt{7} = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$, $q \neq 0$ and $\gcd(p, q) = 1$. This implies that

$$n\sqrt{7} = \frac{p}{q} - m = \frac{p - mq}{q}$$

and (since $n \neq 0$), $\sqrt{7} = \dfrac{p - mq}{nq}$. But this a rational expression, which contradicts that $\sqrt{7}$ is irrational. Hence we have $m + n\sqrt{7}$ as irrational for $m, n \in \mathbb{Q}$, $n \neq 0$.

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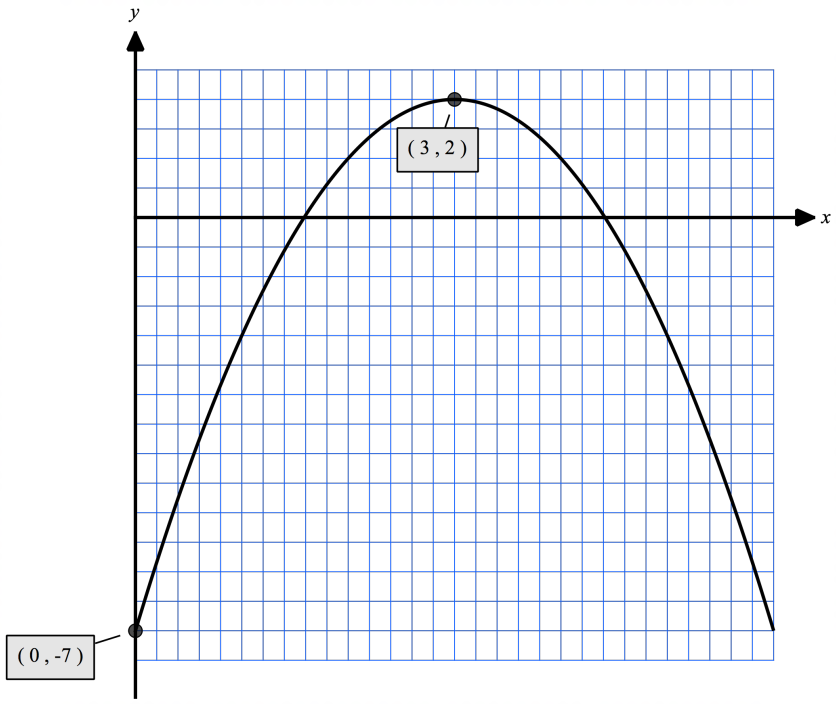
Question Sheet: **Sheet 9**

Model Solution No: 3

(a) **Answer:** $f(x) = 2 - (x-3)^2$

(b) **Answer:**



(c) **Answer:** the gradient at $x = 3$ is zero/the tangent line at $x = 3$ does not intersect the x axis

(d) $f'(x) = 6 - 2x$. Then N-R method gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{6(1) - (1)^2 - 7}{6 - 2(1)} = 1.5$$

We want to use the method twice so

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{2} - \frac{6(3/2) - (3/2)^2 - 7}{6 - 2(3/2)} = 1.5833...$$

Hence $\alpha = 1.5833$ to 4 dp

Answer: 1.5833

(e) Solve the quadratic equation $f(x) = 0$ directly to find the exact value of α. Since $0 < \alpha < 3$, you know that α is the smaller solution. Details omitted.

Answer: $\alpha = 3 - \sqrt{2}$

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Question Sheet: **Sheet 9**

Model Solution No: 4

- First find A. This is where the curves intersect, so the x coordinate of A satisfies $8x\mathrm{e}^{-2x} = 2x$
- Re-arranging this equation is equivalent to $x(8\mathrm{e}^{-2x} - 2) = 0$
- Now $x_A \neq 0$, so $8\mathrm{e}^{-2x} - 2 = 0$. Thus $\mathrm{e}^{-2x} = \frac{1}{4}$ and $-2x = \ln\frac{1}{4}$.
- This gives $x = -\frac{1}{2}\ln\frac{1}{4} = \ln 2$ (remembering to keep exact values throughout)
- Then the area of interest will be found by integrating the curve from 0 to $\ln 2$ and then subtracting the area of the triangle.
- Area of the curve. This is given by (and evaluated by parts)

$$\int_0^{\ln 2} 8x\mathrm{e}^{-2x}\,\mathrm{d}x = 8x\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\Bigg|_0^{\ln 2} + 4\int_0^{\ln 2}\mathrm{e}^{-2x}\,\mathrm{d}x$$

  which if you evaluate and sub in all the limits you get $\frac{3}{2} - \ln 2$.
- Now the triangle has base $\ln 2$ and base $2\ln 2$. Thus its area is $(\ln 2)^2$ (NB this is NOT $\ln 4$)
- Hence our final area is $\frac{3}{2} - \ln 2 - (\ln 2)^2$.

Answer: $\frac{3}{2} - \ln 2 - (\ln 2)^2$

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Question Sheet: **Sheet 9**

Model Solution No: 5

(a) **Answer:** 54400

(b) We want to solve

$$\begin{aligned}
&112 - 0.4(n-12)^2 > 63.6 \\
&\Rightarrow 112 - 0.4(n^2 - 24n + 144) > 63.6 \\
&\Rightarrow -0.4n^2 + 9.6n - 9.2 > 0 \\
&\Rightarrow 0.4n^2 - 9.6n + 9.2 < 0 \\
&\Rightarrow n^2 - 24n + 23 < 0 \\
&\Rightarrow (n-23)(n-1) < 0 \\
&\Rightarrow 1 < n < 23
\end{aligned}$$

Hence the length of time for which the population is greater than 63000 is 22 hours.

Answer: 22 hours

(c) $T(3) - T(2) = [112 - 0.4(3-12)^2] - [112 - 0.4(2-12)^2] = 7.6$

So the change in population is 7600

(d) **Answer:** e.g. $n = 12$ corresponds to time where the size of the population is largest

(e) Need to find the time(s) at which $T = 0$. This is from solving the equation

$$\begin{aligned}
&112 - 0.4(n-12)^2 = 0 \\
&\Rightarrow (n-12)^2 = 280 \\
&\Rightarrow n - 12 = \pm 2\sqrt{70} \\
&\Rightarrow n = 12 \pm 2\sqrt{70}
\end{aligned}$$

We need to the larger the two solutions, so the bacteria die out when $n = 12 + 2\sqrt{70}$.

The antibiotic is administrated at $n = 12$, so the time taken for the antibiotic to eliminate the colony is $12 + 2\sqrt{70} - 12 = 2\sqrt{70} = 16.733...$ hours.

Answer: 16.7 hours (3 sf)

(e) **Answer:** e.g. the bacteria are unlikely to react instantly to the antibiotic / there will be a time delay between administration and first signs of die out

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Question Sheet: **Sheet 9**

Model Solution No: 6

(a) Consider the whole system. Then

$$5400 - 750 - 500 = (2400 + 1000)a$$

which gives the acceleration of the system $a = 1.2205...$

Answer: $a = 1.22$ m s^{-2}

(b) This is a situation for SUVAT. The SUVAT variables are $s = X$, $u = 0$, $v =?$, $a = 1.2205...$, $t = 30$.

Thus $v = u + at = 0 + (1.2205...)(30) = 36.61...$ m s^{-1}

Answer: speed $= 36.6$ m s^{-1} (3 sf)

(c) Consider particle Q (but you can equally consider P) and get

$$5400 - T - 500 = 2400(1.2205...)$$

and this gives $T = 1970.58...$

$T = 1970$ (3 sf) [FYI the equation of motion for P is $T - 750 = 1000(1.2205...)$]

(d) When F is removed, the acceleration is given by

$$-750 - 500 = (2400 + 1000)a$$

which gives $a = -0.3676....$

The time taken to come to rest is solved using SUVAT again. The key equation here is $v = u + at$ with $v = 0$, $u = 36.61...$ (part b) and $a = -0.3676....$ Substituting in and re-arranging for t gives $t = 99.59...$ s

Answer: $t = 99.6$ s (3 sf)

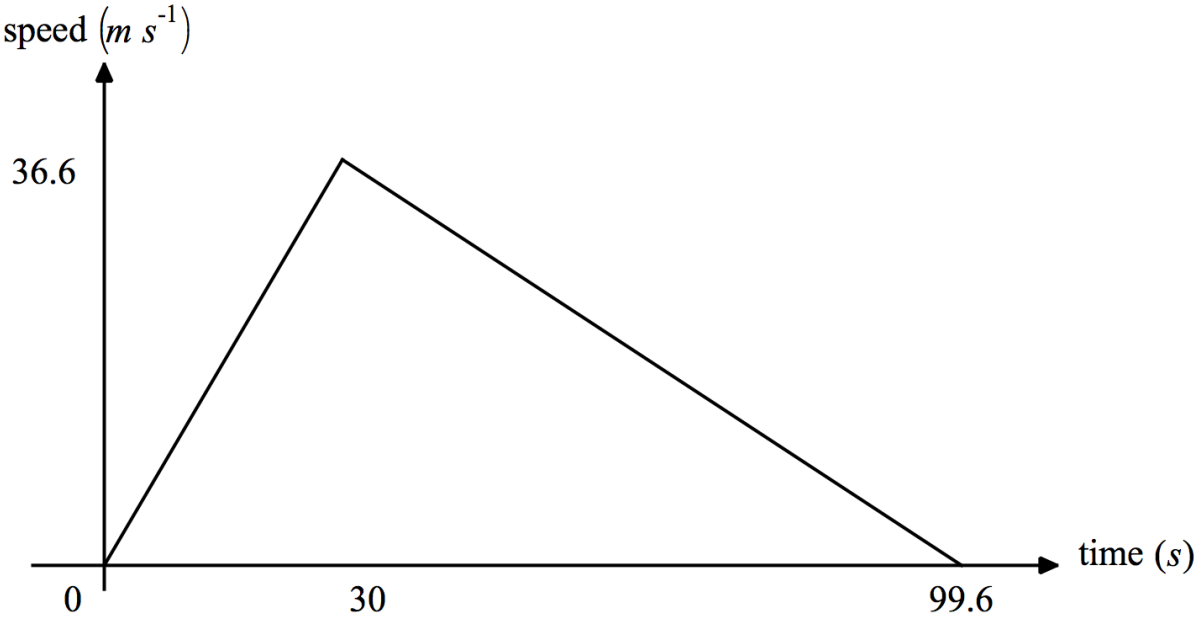
(e) **Solution:** Consider particle Q, then

$$-500 - T = 2400(-0.3675...)$$

which gives $T = -382$ N. The key thing here is that $T > 0$ which means it does indeed act in the same direction as the resisitive force, i.e. into the rod. Hence the rod is in tension.

NB the direction you choose initially doesn't matter. You will either get a positive sign or a negative sign and that tells you if your chosen direction for the force in the rod is correct. In our case, we find that the force in the rod acts in the direction QP so it is a tension.

(f)



(g) **Answer:** e.g. a driving force (produced by the caravan)

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Question Sheet: **Sheet 9**

Model Solution No: 7

(a) **Answer:** e.g. Generate a list of random numbers. Pick data points from the large data set that corresponds to the random number. If the random number is out of range, then ignore it and use the next one. If the random number repeats, ignore it and use the next one. Continue in this way until a sample of size 12 has been selected.

(b) **Answer:** e.g. small amounts of rainfall are indicated as 'tr' which is not numerical

(c) **Answer:** e.g. for every 1 mm increase in rainfall, the pressure decreases by 0.4136 hPa. [Note units are necessary!]

(d) (i) **Answer:** most of the values of r in the data set are zero/close to 0 (and the regression line predicts a pressure of 1012 hPa at $r = 0$)

(ii) **Answer:** e.g. she has only used 12 data points which may not take into account monthly/seasonal variations

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