IYGB - FP1 PAPER P - QUESTION 1

TIDY IN STAGES

$$\Rightarrow z = (2-i)^2 + \frac{7-4i}{2+i} - 8$$

$$\Rightarrow z = 2^2 - 2\times 2\times i + (i)^2 + \frac{(7-4i)(2-i)}{(2+i)(2-i)} - 8$$

$$\Rightarrow z = 4 - 4i - 1 + \frac{14 - 7i - 8i + 4i^2}{4 \cancel{-2i} \cancel{+2i} - i^2} - 8$$

$$\Rightarrow z = 3 - 4i + \frac{14 - 15i - 4}{4 + 1} - 8$$

$$\Rightarrow z = 3 - 4i + \frac{10 - 15i}{5} - 8$$

$$\Rightarrow z = 3 - 4i + 2 - 3i - 8$$

$$\Rightarrow z = -3 - 7i$$

IYGB - FP1 PAPER P - QUESTION 2

USING STANDARD SUMMATION FORMULAE

$$\sum_{r=1}^{2n}\left(3r^2-\frac{1}{2}\right) = 3\sum_{r=1}^{2n} r^2 - \frac{1}{2}\sum_{r=1}^{2n} 1$$

$$= \underbrace{3\times\frac{1}{6}(2n)\left[(2n)+1\right]\left[2(2n)+1\right]}_{\boxed{\sum_{r=1}^{n} r^2 \equiv \frac{1}{6}n(n+1)(2n+1)}} - \frac{1}{2}\times 2n$$

$$= n(2n+1)(4n+1) - n$$

$$= n\left[(2n+1)(4n+1) - 1\right]$$

$$= n\left[8n^2+6n+1-1\right]$$

$$= n\left(8n^2+6n\right)$$

$$= 2n^2(4n+3)$$

IYGB - FP1 PAPER P - QUESTION 3

a) APPLYING A ROW OPERATION $r_{13}(-1)$ YIELDS ZERO ROW AT $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}$$

ON MULTIPLICATION: $\underline{A}\underline{B} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$, SO A SQUARE

MATRIX WITH A ZERO ROW (OR COLUMN) HAS ZERO DETERMINANT

b) (NOTE THAT THE CONVERSE IS NOT TRUE DUE TO THE WAY MATRICES MULTIPLY)

$$\underline{B}\underline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ k & k+4 \end{pmatrix}$$

$\det(\underline{B}\underline{A}) = 8 \neq 0$ FOR ALL k, SO NON SINGULAR

c) IF $k=-2$

$$\underline{B}\underline{A} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2\underline{I}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} = \sqrt{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

→ STANDARD MATRIX ROTATION, CLOCKWISE ABOUT O, BY 45°

UNIFORM ENLARGEMENT ABOUT O, WITH SCALE FACTOR $2\sqrt{2} = \sqrt{8}$ $\therefore a = 8$

IYGB - FP1 PAPER P - QUESTION 4

a) AS THE COEFFICIENTS OF THE POLYNOMIAL EQUATION ARE REAL, ANY COMPLEX ROOTS MUST APPEAR AS CONJUGATE PAIRS — SO WE HAVE

$$z_1 = 1+2i \quad , \text{ SAY } \alpha$$
$$z_2 = 1-2i \quad , \text{ SAY } \beta$$

NOW $\alpha+\beta+\gamma = -\frac{b}{a}$

$$(1+2i)+(1-2i)+\gamma = -\frac{-1}{2}$$
$$2+\gamma = \frac{1}{2}$$
$$\gamma = -\frac{3}{2}$$

$\therefore$ SOLUTIONS ARE $1+2i$, $1-2i$ & $-\frac{3}{2}$

b) NOW $\alpha\beta\gamma = -\frac{d}{a}$

$$(1+2i)(1-2i)\left(-\frac{3}{2}\right) = -\frac{p}{2}$$
$$3(1+2i)(1-2i) = p$$
$$p = 3(1^2+2^2)$$
$$p = 15$$

ALTERNATIVE WITHOUT USING ROOT RELATIONSHIPS

$$(1+2i)^2 = 1+4i+(2i)^2 = 1+4i-4 = -3+4i$$
$$(1+2i)^3 = (-3+4i)(1+2i) = -3-6i+4i-8 = -11-2i$$

SUB INTO THE CUBIC TO FIND p FIRST

$$2z^3 - z^2 + 4z + p = 0$$
$$2(-11-2i)-(-3+4i)+4(1+2i)+p = 0$$
$$-22-\cancel{4i}+3-\cancel{4i}+4+\cancel{8i}+p=0$$
$$p = 15$$

IYGB – FP1 PAPER P – QUESTION 4

NOW SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS IF COMPLEX

$$(z-1-2i)(z-1+2i) = [(z-1)-2i][(z-1)+2i]$$
$$= (z-1)^2 - (2i)^2$$
$$= z^2 - 2z + 1 + 4$$
$$= z^2 - 2z + 5$$

BY INSPECTION

$$2z^3 - z^2 + 4z + 15 = (2z+3)(z^2-2z+5)$$

$$\therefore \ z = \begin{cases} 1+2i \\ 1-2i \\ -\frac{3}{2} \end{cases}$$

IYGB - FP1 PAPER P - QUESTION 5

- WRITE THE EQUATIONS OF THE PLANES IN MATRIX FORM

$$\left.\begin{aligned} x - 3y - 2z &= 2 \\ 2x - 2y + 3z &= 1 \\ 5x - 7y + 4z &= k \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 2 & -2 & 3 \\ 5 & -7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ k \end{bmatrix}$$

- CHECK IF A UNIQUE SOLUTION EXISTS (IN TERMS OF k)

$$\begin{vmatrix} 1 & -3 & -2 \\ 2 & -2 & 3 \\ 5 & -7 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ -7 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ 5 & -7 \end{vmatrix}$$

$$= (-8+21) + 3(8-15) - 2(-14+10)$$

$$= 13 - 21 + 8$$

$$= 0$$

NO UNIQUE SOLUTION EXISTS

- WRITE THE SYSTEM AS AN AUGMENTED MATRIX & ROW REDUCE

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 2 & -2 & 3 & 1 \\ 5 & -7 & 4 & k \end{array}\right] \underset{r_{13}(-5)}{\overset{r_{12}(-2)}{\Longrightarrow}} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 4 & 7 & -3 \\ 0 & 8 & 14 & k-10 \end{array}\right]$$

$$\overset{r_2\left(\frac{1}{4}\right)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 8 & 14 & k-10 \end{array}\right] \overset{r_{23}(-8)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 2 \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & k-4 \end{array}\right]$$

- FOR A SOLUTION $k=4$

$$\overset{r_{21}(3)}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{7}{4} & -\frac{3}{4} \end{array}\right]$$

IYGB – FP1 PAPER P – QUESTION 5

- **EXTRACTING A SOLUTION OUT**

$$\left.\begin{aligned} x + \tfrac{13}{4}z &= -\tfrac{1}{4} \\ y + \tfrac{7}{4}z &= -\tfrac{3}{4} \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= -\tfrac{1}{4} - \tfrac{13}{4}z \\ y &= -\tfrac{3}{4} - \tfrac{7}{4}z \end{aligned}$$

- **LET $z = -4t - 1$**

$$\left.\begin{aligned} x &= -\tfrac{1}{4} - \tfrac{13}{4}(-4t-1) \\ y &= -\tfrac{3}{4} - \tfrac{7}{4}(-4t-1) \\ z &= -4t-1 \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= -\tfrac{1}{4} + 13t + \tfrac{13}{4} \\ y &= -\tfrac{3}{4} + 7t + \tfrac{7}{4} \\ z &= -4t-1 \end{aligned}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 + 13t \\ 1 + 7t \\ -1 - 4t \end{pmatrix} \quad //$$

IYGB - FP1 PAPER P - QUESTION 6

- USING THE STANDARD RELATIONSHIPS BETWEEN THE ROOTS AND THE COEFFICIENTS OF A CUBIC

$$x^3 + 2x^2 + 10x + k = 0$$

$$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{-2}{1} = -2$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = \frac{10}{1} = 10$$

$$\alpha\beta\gamma = -k$$

- AS THE ROOTS ARE IN GEOMETRIC PROGRESSION

$$\alpha+\beta+\gamma = \alpha + \alpha r + \alpha r^2$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = \alpha(\alpha r) + (\alpha r)(\alpha r^2) + (\alpha r^2)\alpha = \alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2$$

$$\alpha\beta\gamma = \alpha(\alpha r)(\alpha r^2) = \alpha^3 r^3$$

- TIDYING UP THESE EXPRESSIONS

$$\left.\begin{array}{l} \alpha + \alpha r + \alpha r^2 = -2 \\ \alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2 = 10 \\ \alpha^3 r^3 = -k \end{array}\right\} \Longrightarrow \begin{array}{rll} \alpha(1+r+r^2) &= -2 & -\text{I} \\ \alpha^2 r(1+r^2+r) &= 10 & -\text{II} \\ k &= -(\alpha r)^3 & -\text{III} \end{array}$$

- DIVIDING EQUATIONS I & II

$$\frac{\alpha^2 r\cancel{(1+r+r^2)}}{\alpha\cancel{(1+r+r^2)}} = \frac{10}{-2} \quad \therefore \quad \alpha r = -5$$

- HENCE EQUATION III GIVES

$$k = -(\alpha r)^3 = -(-5)^3 = 125$$

IYGB - FP1 PAPER P - QUESTION 7

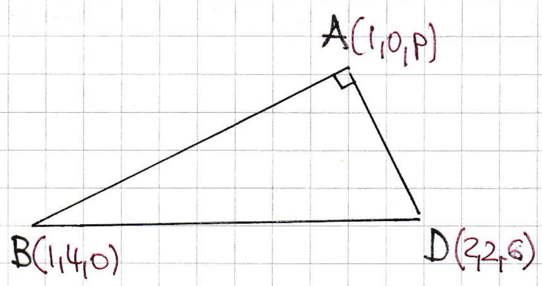
a) FIND A DIRECTION VECTOR FOR L

$$\overrightarrow{BD} = \underline{d} - \underline{b} = (2,2,6) - (1,4,0) = (1,-2,6)$$

$$\underline{r} = (1,4,0) + \lambda(1,-2,6)$$

$$(x,y,z) = (\lambda+1,\ 4-2\lambda,\ 6\lambda)$$

b) LOOKING AT THE DIAGRAM



- $\overrightarrow{AB} = \underline{b} - \underline{a} = (1,4,0) - (1,0,p) = (0,4,-p)$
- $\overrightarrow{AD} = \underline{d} - \underline{a} = (2,2,6) - (1,0,p) = (1,2,6-p)$

$$\Rightarrow \overrightarrow{AB} \cdot \overrightarrow{AD} = 0$$

$$\Rightarrow (0,4,-p)\cdot(1,2,6-p) = 0$$

$$\Rightarrow 0 + 8 - p(6-p) = 0$$

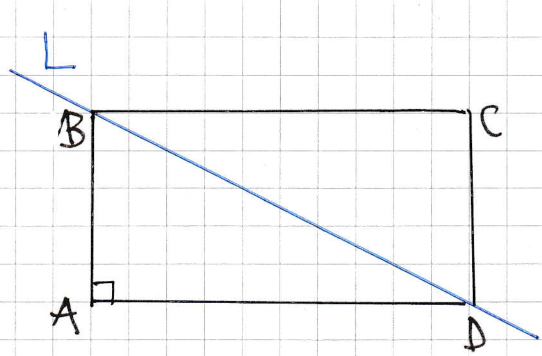
$$\Rightarrow 8 - 6p + p^2 = 0$$

$$\Rightarrow p^2 - 6p + 8 = 0$$

$$\Rightarrow (p-4)(p-2) = 0$$

$$p = \begin{cases} 2 \\ 4 \end{cases}$$

c) LOOKING AT THE RECTANGLE



- AREA $\square ABCD = 12\sqrt{2}$

  (AREA $\triangle BAD = 6\sqrt{2}$)

- IF $p = 2$

$$|AB||AD| = |(0,4,-2)||(1,2,4)| = \sqrt{16+4}\ \sqrt{1+4+16} \neq 12\sqrt{2}$$

IYGB – FP1 PAPER P – QUESTION 7

- IF $p = 4$

$$|AB||BD| = |0,4,-4||1,2,2| = \sqrt{16+16}\sqrt{1+4+4} = \sqrt{32} \times 3$$
$$= 12\sqrt{2}$$

$\therefore A(1,0,4)$

HENCE BY INSPECTION

$$\begin{tikzcd}
B(1,4,0) \arrow[rr, "(+1,+2,+2)"] & & C(2,6,2) \\
A(1,0,4) \arrow[u, "{[+0,+4,-4]}"'] \arrow[rr, "(+1,+2,+2)"'] & & D(2,2,6) \arrow[u, "{[+0,+4,-4]}"']
\end{tikzcd}$$

$\therefore C(2,6,2)$

IYGB - FP1 PAPER P - QUESTION 8

BASE CASE $n=1$

- $\displaystyle\sum_{r=1}^{1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 1\times2\times\left(\tfrac{1}{2}\right)^{0} = 2$

- $16-\left(\tfrac{1}{2}\right)^{n-1}\left(n^2+5n+8\right) = 16-\left(\tfrac{1}{2}\right)^{0}\times(1+5+8) = 16-1\times14 = 2$

I.E THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k,\ k\in\mathbb{N}$

$$\Rightarrow \sum_{r=1}^{k}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16-\left(\tfrac{1}{2}\right)^{k-1}\left(k^2+5k+8\right)$$

$$\Rightarrow \sum_{r=1}^{k}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] + (k+1)(k+2)\left(\tfrac{1}{2}\right)^{k} = 16-\left(\tfrac{1}{2}\right)^{k-1}\left(k^2+5k+8\right) + (k+1)(k+2)\left(\tfrac{1}{2}\right)^{k}$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16+\left(\tfrac{1}{2}\right)^{k}(k+1)(k+2)-\left(\tfrac{1}{2}\right)^{k-1}\left(k^2+5k+8\right)$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16+\left(\tfrac{1}{2}\right)^{k}\left[(k+1)(k+2)-\left(\tfrac{1}{2}\right)^{-1}\left(k^2+5k+8\right)\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16+\left(\tfrac{1}{2}\right)^{k}\left[k^2+3k+2-2\left(k^2+5k+8\right)\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16+\left(\tfrac{1}{2}\right)^{k}\left[k^2+3k+2-2k^2-10k-16\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16+\left(\tfrac{1}{2}\right)^{k}\left[-k^2-7k-14\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16-\left(\tfrac{1}{2}\right)^{k}\left(k^2+7k+14\right)$$

IYGB - FP1 PAPER P - QUESTION 8

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16 - \left(\tfrac{1}{2}\right)^{k}\left[\underbrace{(k^2+2k+1)+5(k+1)+8}_{k^2+7k+14}\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16 - \left(\tfrac{1}{2}\right)^{k}\left[(k+1)^2+5(k+1)+8\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\tfrac{1}{2}\right)^{r-1}\right] = 16 - \left(\tfrac{1}{2}\right)^{(k+1)-1}\left[(k+1)^2+5(k+1)+8\right]$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$ — SINCE THE RESULT HOLDS FOR $n=1$ THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$