CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 5**

Model Solution No: 1

First re-arrange the equation

$$\begin{aligned}
&\frac{x^2-4x-4}{x+1}=k\\
&\Rightarrow x^2-4x-4=k(x+1)\\
&\Rightarrow x^2-4x-4-kx-k=0\\
&\Rightarrow x^2+x(-4-k)+(-4-k)=0
\end{aligned}$$

No real solutions means the discriminant of this is negative:

$$\begin{aligned}
b^2-4ac<0 &\Rightarrow (-4-k)^2-4(1)(-4-k)<0\\
&\Rightarrow k^2+8k+16+16+4k<0\\
&\Rightarrow k^2+12k+32<0
\end{aligned}$$

The critical values of this inequality are when $k^2+12k+32=0$. The equation factorises to $(k+8)(k+4)=0$ and thus the critical values are $k=-4$ and $k=-8$.

Hence the values for which $k^2+12k+32<0$ are $-8<k<-4$ (convince yourself by a graph sketch, for example). The final step is to translate this into set notation.

Answer: $\{k\in\mathbb{R}:-8<k<-4\}$

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Question Sheet: **Sheet 5**

Model Solution No: 2

(a) **Answer:** Error 1: the expansion of $\cos(x+h)$ is incorrect. It contains a plus sign between the terms, when it should actually be a minus sign. Error 2: the small angle approximation for $\cos(h)$ is not h.

(b) **Solution:**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\left(1-\frac{h^2}{2}\right)-\sin x(h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-\frac{h^2}{2}\cos x-h\sin x}{h} \\
&= \lim_{h\to 0}\left(-\frac{h}{2}\cos x-\sin x\right) \\
&= -\sin x
\end{aligned}$$

Question Sheet: **Sheet 5**

Model Solution No: 3

(a) **Answer:** 17.978, 28.419

(b) Using the trapezium rule with a strip width of e gives:

$$\int_{\mathrm{e}}^{5\mathrm{e}} (5x - x\ln x)\,\mathrm{d}x \approx \frac{\mathrm{e}}{2}\left[10.873 + 32.491 + 2(17.978 + 23.660 + 28.419)\right]$$

which gives an approximation of 249.37 to two decimal places.

Answer: 249.37 units^2

(c) Use the distributive properties of the integral to see that:

$$\int_{\mathrm{e}}^{5\mathrm{e}} (5x - x\ln x + 2)\,\mathrm{d}x = \int_{\mathrm{e}}^{5\mathrm{e}} (5x - x\ln x)\,\mathrm{d}x + \int_{\mathrm{e}}^{5\mathrm{e}} 2\,\mathrm{d}x$$

The first integral we already have an estimate for from part (a). The second we can calculate easily as

$$\int_{\mathrm{e}}^{5\mathrm{e}} 2\,\mathrm{d}x = [2x]_{\mathrm{e}}^{5\mathrm{e}} = 2(5\mathrm{e}) - 2\mathrm{e} = 8\mathrm{e}$$

Thus our approximation for this will be 249.3724... + 8e = 271.12 to two decimal places (using the unrounded version from (a))

Answer: 271.12

Question Sheet: **Sheet 5**

Model Solution No: 4

(a) **Solution:** Re-arranging, we have

$$(m-p)\mathbf{a} = (q-n)\mathbf{b}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel, this equality cannot happen unless $m-p=0$ and $q-n=0$. Thus $m=p$ and $q=n$

OR re-arranging gives

$$(m-p)\mathbf{a} + (n-q)\mathbf{b} = 0$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel, we need both $m-p=0$ and $q-n=0$ so that we can combine the terms and get 0. Thus $m=p$ and $q=n$

(b) **Answer:** $\overrightarrow{AB} = 5\mathbf{i} + (-1-s)\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{BC} = (s-6)\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

Since they have equal magnitudes, we must have (after squaring both sides in the second line):

$$\begin{aligned}
&\sqrt{5^2 + (-1-s)^2 + 3^2} = \sqrt{(s-6)^2 + 4^2 + 5^2} \\
&\Rightarrow 25 + 1 + 2s + s^2 + 9 = s^2 - 12s + 36 + 16 + 25 \\
&\Rightarrow 14s = 42 \\
&\Rightarrow s = 3
\end{aligned}$$

Answer: $s = 3$

Question Sheet: **Sheet 5**

Model Solution No: 5

(a) **Answer:** Initial position is when $t = 0$, so $(1, 2)$. Final position is when $t = 3\pi$, so $(1, -2)$

(b) (i) **Solution:** $x - 1 = -2\sin t$ and $y = 2\cos t$. Squaring and adding both equations we get $(x-1)^2 + y^2 = 4\sin^2 t + 4\cos^2 t$ and thus $(x-1)^2 + y^2 = 4$, which describes a circle.

(ii) Centre of the circle is $(1, 0)$ and the radius is 2

(c) **Answer:** the robot is moving anticlockwise around the circle

...because (e.g.) the robot starts at the top of the circle at $(1, 2)$ and after a quarter of a revolution ($t = \frac{\pi}{2}$), the robot is located at $(-1, 0)$, so to the left.

NB: there are many ways to think about part (c). The best way to get this is draw a small sketch, plug in values of t to see in which direction the parametric equations trace out the path. It is useful to remember that $t = 2\pi$ corresponds to a full revolution, $t = \pi$ corresponds to half a revolution, etc.

(d) $0 \le t \le 8\pi$

Question Sheet: **Sheet 5**

Model Solution No: 6

(a) You need clear, systematic working for equations like this. Start by considering each particle separately to obtain their equations of motions.

Considering A, Newton's second law considered parallel to the plane (i.e. resolving parallel to the plane) gives:

$$4mg\sin 30 - \mu R_A - T = 4ma$$

Now $\mu = \frac{\sqrt{3}}{10}$ and by resolving perpendicular to the plane, we have $R_A = 4mg\cos 30$. Plugging this in and re-arranging gives us

$$T + 4am = \frac{7mg}{5} \qquad (1)$$

Now we do the same thing for B. Resolving parallel to the plane and in its direction of motion, we get

$$T - mg\sin 30 - \mu R_B = ma$$

Again $\mu = \frac{\sqrt{3}}{10}$ and here we have $R_B = mg\cos 30$ by resolving perpendicular to the plane. Substituting all this in and doing some re-arranging, we get the equation of motion for B as:

$$T - am = \frac{13mg}{20} \qquad (2)$$

Now we have two simultaneous equations which we can solve for a and T. Subtracting the equations gives:

$$5am = \frac{3}{4}mg$$

and thus $a = \dfrac{3}{20}g$. Then substituting this back into (2) (for example), we get that the tension is $T = \dfrac{4}{5}mg$.

Answer: $T = \dfrac{4}{5}mg$, $a = \dfrac{3}{20}g$

(b) Both strings act an angle of 60° relative to the vertical passing through the pulley. The resultant forced by both strings is thus

$$2 \times \frac{4mg}{5}\cos 60 = \frac{4mg}{5}$$

and the direction is vertically downwards.

Extension (you will need to fill in some of these gaps):

- After 5 seconds, the speed of B is $\frac{3}{4}g$ (using $v = u + at$)
- After 5 seconds, B has a travelled a distance of $\frac{15}{8}g$
- After the string breaks, the deceleration a on B is given by $ma = mg\sin 30 + \frac{\sqrt{3}}{10}(mg\cos 30)$ and so the deceleration of B is $\frac{13}{20}g$
- Hence B comes to rest after travelling a distance of $\frac{45}{104}g$ up the plane (using $v^2 = u^2 + 2as$)
- Now B is at it highest point above the ground. It will accelerate down the plane at $\frac{\sin 30 - \frac{\sqrt{3}}{10}\cos 30}{m} = \frac{7}{20}g$ and it will need to travel a total distance of $\frac{45}{104}g + \frac{15}{8}g + \frac{6}{\sin 30} = \frac{450}{13}$ m
- Using $s = ut + \frac{1}{2}at^2$ with our found values, it will take

$$t = \sqrt{\frac{2(450/13)g}{7g/20}} = 1.5724...$$

Answer: $t = 1.5$ s (2 sf)

If you want, now have a go at finding the time from the point when A is released from rest. It's just one more calculation and some addition. But we omit this here.

Question Sheet: **Sheet 5**

Model Solution No: 7

(a) (i) **Answer:** e.g. both located at similar distances from the equator / similar latitudes

(a) (ii) **Answer:** e.g. the IQR for Jacksonville is 2^o C and the IQR for Beijing is 3^o, which are close/similar (NB: references to median is B0 because this tells you about location, not spread)

(a) (iii) **Answer:** e.g. it is affected by outliers/extreme values

(b) **Solution** For Beijing, the IQR is 3. So $1.5 \times 3 = 4.5$. The lower outlier boundary is $25 - 4.5 = 20.5$. Upper outlier boundary is $28 + 4.5 = 32.5$. No data below 20.5, but there is at least one value in the data above 32.5 (since the max value is 33). Hence the Beijing data contains at least one outlier (condone 'contains outliers').

For Jacksonville, the IQR is 2. So $1.5 \times 2 = 3$. The lower outlier boundary is $26 - 3 = 23$. The upper outlier boundary is $28 + 3 = 31$. No data below 23 or above 31, so no outliers in the Jacksonville data according to this measure.

For Perth, the IQR is 4.5. So $1.5 \times 4.5 = 6.75$. The lower outlier boundary is $11 - 6.75 = 4.25$. The upper outlier boundary is $15.5 + 6.75 = 22.25$. No data below 4.25 or above 22.25, so no outliers in the Perth data according to this measure.

(c) **Answer:** expect a similar temperature profile to Perth, so suggest average temperature equal to the median for Perth = 13^o C.

Extension: e.g. expect higher median temperature and higher quartiles because summer starts in December in Perth/it is winter in Perth in July

Expect a similar spread in the data (or similar IQR/range) because it is close to the sea (so expect the same sort of variation/equable climate)

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