IYGB - FP1 PAPER V - QUESTION 1

WE ARE GIVEN THAT $|\underline{a}| = |\underline{b}| = 1$

THEN WE HAVE $\underline{p} \perp \underline{q}$, IF $\underline{p} \cdot \underline{q} = 0$

$$\Rightarrow (\underline{a} + 2\underline{b}) \cdot (5\underline{a} - 4\underline{b}) = 0$$

$$\Rightarrow 5\underline{a} \cdot \underline{a} - 4\underline{a} \cdot \underline{b} + 10\underline{a} \cdot \underline{b} - 8\underline{b} \cdot \underline{b} = 0$$

$$\Rightarrow 5|\underline{a}||\underline{a}|\cos 0 + 6\underline{a} \cdot \underline{b} - 8|\underline{b}||\underline{b}|\cos 0 = 0$$

$$\Rightarrow 5 \times 1 \times 1 \times 1 + 6\underline{a} \cdot \underline{b} - 8 \times 1 \times 1 \times 1 = 0$$

$$\Rightarrow 6\underline{a} \cdot \underline{b} = 3$$

$$\Rightarrow \underline{a} \cdot \underline{b} = \frac{1}{2}$$

$$\Rightarrow |\underline{a}||\underline{b}|\cos\theta = \frac{1}{2}$$

$$\Rightarrow 1 \times 1 \times \cos\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = 60°$$

IYGB - MP1 PAPER V - QUESTION 2

a) STARTING WITH A DIAGRAM

$\arg(17+ki-1-i)=\theta$

$\arg(16+(k-1)i)=\theta$

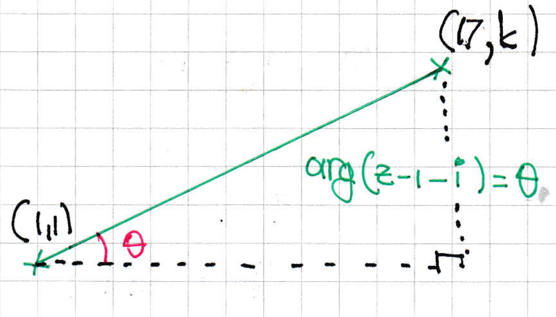
$\arctan\left(\frac{k-1}{16}\right)=\theta$

$\arctan\left(\frac{k-1}{16}\right)=\arctan\frac{3}{4}$

$\frac{k-1}{16}=\frac{3}{4}$

$4k-4=48$

$k=13$

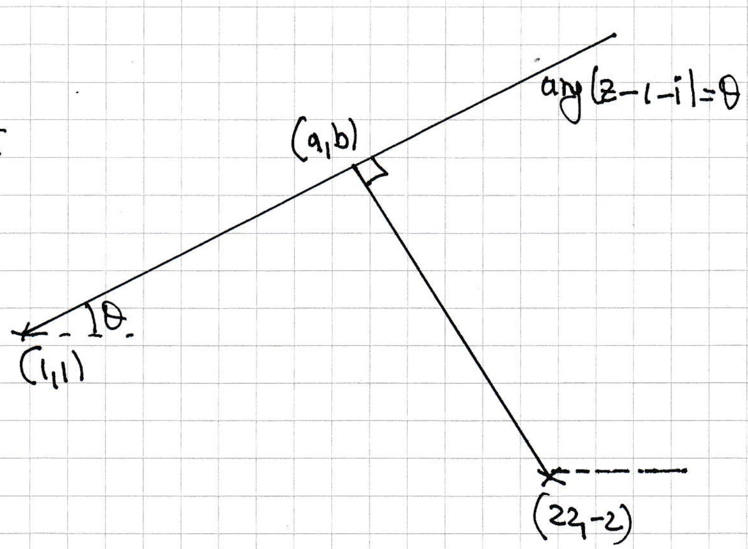


(or simple trigonometry on the above triangle)

b) NOW SUPPOSE THE REQUIRED COMPLEX NUMBER IS $a+bi$

- IF $|z-22+2i|$ IS TO BE LEAST WE MUST HAVE A RIGHT ANGLE

- GRADIENTS MUST BE THE NEGATIVE RECIPROCALS OF EACH OTHER



HENCE WE HAVE

$\frac{b-1}{a-1}=\frac{3}{4}$ & $\frac{b+2}{a-22}=-\frac{4}{3}$

$4b-4=3a-3$ $\qquad$ $3b+6=-4a+88$

$4b=3a+1$ $\qquad$ $3b=-4a+82$

$12b=9a+3$ $\qquad$ $12b=-16a+328$

IYGB – FP1 PAPER QUESTION

$\Rightarrow 9a + 3 = -16a + 328$

$\Rightarrow 25a = 325$

$\Rightarrow a = 13$

&

$4b = 3a + 1$

$4b = 40$

$b = 10$

$\therefore 13 + 10i$

IYGB - FP1 PAPER V - QUESTION 3

$$f(n) = 2^n + 6^n \quad , \quad n \in \mathbb{N}$$

BASE CASE

$f(1) = 2^1 + 6^1 = 2 + 6 = 8$, IE DIVISIBLE BY 8

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, IE $f(k) = 8m$, $m \in \mathbb{N}$.

$$\Rightarrow f(k+1) - f(k) = \left[2^{k+1} + 6^{k+1}\right] - \left[2^k + 6^k\right]$$

$$\Rightarrow f(k+1) - 8m = 2^{k+1} - 2^k + 6^{k+1} - 6^k$$

$$\Rightarrow f(k+1) - 8m = 2 \times 2^k - 2^k + 6 \times 6^k - 6^k$$

$$\Rightarrow f(k+1) - 8m = 2^k + 5 \times 6^k$$

$$\Rightarrow f(k+1) - 8m = \left[f(k) - 6^k\right] + 5 \times 6^k$$

$$f(k) = 2^k + 6^k$$
$$2^k = f(k) - 6^k$$

$$\Rightarrow f(k+1) - 8m = f(k) + 4 \times 6^k$$

$$\Rightarrow f(k+1) - 8m = 8m + 4 \times 6 \times 6^{k-1}$$

$$\Rightarrow f(k+1) = 16m + 24 \times 6^{k-1}$$

$$\Rightarrow f(k+1) = 8\left[2m + 3 \times 6^{k-1}\right]$$

CONCLUSION

IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD

IYGB - FP1 PAPER V - QUESTION 4

USING THE STANDARD RESULT FOR VOLUME OF REVOLUTION IN x

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx = \pi \int_1^e (x \ln x)^2 \, dx = \pi \int_1^e x^2 (\ln x)^2 \, dx$$

CONTINUE BY INTEGRATION BY PARTS & IGNORING π & LIMITS

$$\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \int \frac{2}{3} x^2 \ln x \, dx$$

$(\ln x)^2$	$2\ln x \times \frac{1}{x}$
$\frac{1}{3}x^3$	x^2

BY PARTS AGAIN ON THIS INTEGRAL

$$\dots = \frac{1}{3}x^3 (\ln x)^2 - \left[\frac{2}{9} x^3 \ln x - \int \frac{2}{9} x^2 \, dx \right]$$

$$\dots = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9} x^3 \ln x + \int \frac{2}{9} x^2 \, dx$$

$$\dots = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + C$$

$\ln x$	$\frac{1}{x}$
$\frac{2}{9}x^3$	$\frac{2}{3}x^2$

RETURNING TO THE MAIN LINE

$$V = \pi \left[\frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 \right]_1^e$$

$$V = \pi \left[\left(\frac{1}{3}e^3 - \frac{2}{9}e^3 + \frac{2}{27}e^3 \right) - \left(0 - 0 + \frac{2}{27} \right) \right]$$

$$V = \pi \left[\frac{5}{27} e^3 - \frac{2}{27} \right]$$

$$V = \frac{\pi}{27} \left(5e^3 - 2 \right)$$

IYGB - FP1 PAPER U - QUESTION 5

FOR THE GIVEN EQUATION

$$2x^3-4x+1=0$$

- $\alpha+\beta+\gamma = -\frac{b}{a} = 0$
- $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = -2$
- $\alpha\beta\gamma = \frac{1}{2}$

AS IT WILL BE DIFFICULT TO OBTAIN A SIMPLIFIED EXPRESSION WE MAY TRANSFORM "PARTLY"

LET $\boxed{\begin{array}{l} y = x-2 \\ x = y+2 \end{array}}$

DOES FINDING THE SUM OF THE 3 ROOTS OF THE CUBIC $2\left(\frac{1}{y+2}\right)^3 - 4\left(\frac{1}{y+2}\right)+1=0$ ALSO WORK ?

$$\Rightarrow 2(y+2)^3-4(y+2)+1=0$$
$$\Rightarrow 2(y^3+6y^2+12y+8)-4y-8+1=0$$
$$\Rightarrow 2y^3+12y^2+24y+16-4y-7=0$$
$$\Rightarrow 2y^3+12y^2+20y+9=0$$

LET THE SOLUTIONS OF THIS CUBIC BE A, B & C

$$\Rightarrow A+B+C = -\frac{12}{2} = -6$$
$$\Rightarrow ABC = -\frac{9}{2}$$
$$\Rightarrow AB+BC+CA = \frac{20}{2} = 10$$

HENCE WE HAVE

$$\frac{1}{\alpha-2}+\frac{1}{\beta-2}+\frac{1}{\gamma-2} = \frac{1}{A}+\frac{1}{B}+\frac{1}{C}$$
$$= \frac{BC+AC+AB}{ABC}$$
$$= \frac{10}{-9/2}$$
$$= -\frac{20}{9}$$

IYGB – FP1 PAPER V – QUESTION 6

WRITE THE LINE IN PARAMETRIC & PICK TWO "RANDOM NICE POINTS"

$$\frac{x-4}{1} = \frac{y-3}{3} = \frac{z-2}{-4}$$

$$\underline{r} = (4,3,2) + \lambda(1,3,-4)$$

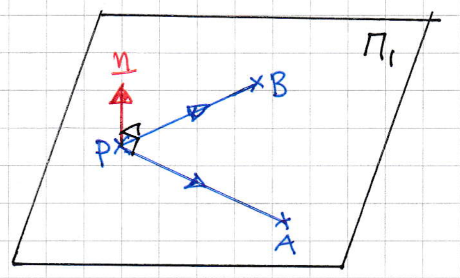
$\therefore$ A(4,3,2) & B(5,6,−2) LIE ON THIS LINE

LOOKING AT THE DIAGRAM

$\vec{PA} = \underline{a} - \underline{p} = (4,3,2) - (1,3,8) = (3,0,-6)$ SCALED TO $(1,0,-2)$

$\vec{PB} = \underline{b} - \underline{p} = (5,6,-2) - (1,3,8) = (4,3,-10)$

LET THE NORMAL BE $\underline{n} = (a,b,c)$



$(1,0,-2) \cdot (a,b,c) = 0$
$(4,3,-10) \cdot (a,b,c) = 0$

$$\left.\begin{array}{l} a - 2c = 0 \\ 4a + 3b - 10c = 0 \end{array}\right\} \Rightarrow \underline{\underline{a = 2c}}$$

$$\Rightarrow 4(2c) + 3b - 10c = 0$$
$$3b - 2c = 0$$
$$\underline{\underline{b = \tfrac{2}{3}c}}$$

LET $c = 3$

THEN $b = 2$ & $a = 6$

$\therefore \underline{n} = (6,2,3)$

THE EQUATION OF THE PLANE IS

$$6x + 2y + 3z = \text{CONSTANT}$$

IYGB - FP1 PAPER V - QUESTION 6

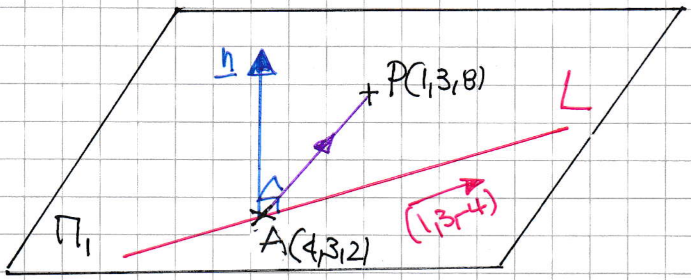
USING $(1,3,8)$

$$(6\times1)+(2\times3)+(3\times8) = \text{CONSTANT}$$

$$\text{CONSTANT} = 36$$

$$\therefore \quad 6x+2y+3z=36$$

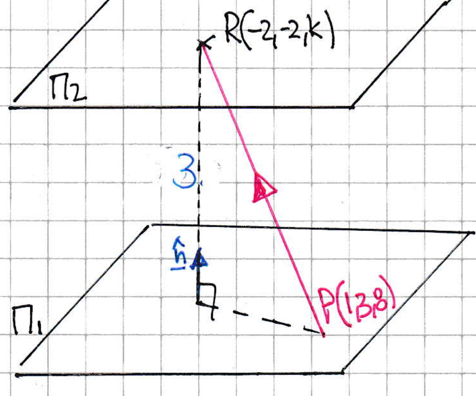
ALTERNATIVE BY CROSS PRODUCT TO FIND THE NORMAL



$$\vec{AP} = \underline{p}-\underline{a} = (1,3,8)-(4,3,2) = (-3,0,6) \quad \text{SCALED TO } (1,0,-2)$$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -2 \\ 1 & 3 & -4 \end{vmatrix} = (6,2,3) \quad \text{AS BEFORE}$$

b) LOOKING AT A DIAGRAM



$$\vec{PR} = \underline{r}-\underline{p} = (-2,-2,k)-(1,3,8)$$
$$= (-3,-5,k-8)$$

IYGB - FP1 PAPER V - QUESTION 6

NEXT WORK THE UNIT NORMAL $\hat{\underline{n}}$

$$\underline{n} = (6,2,3)$$

$$|\underline{n}| = \sqrt{36+4+9} = 7$$

$$\hat{\underline{n}} = \frac{1}{7}(6,2,3)$$

PROJECTING $\overrightarrow{PR}$ ONTO THE UNIT NORMAL $\hat{\underline{n}}$ GIVES 3

$$\Longrightarrow \quad d = |\overrightarrow{PR} \cdot \hat{\underline{n}}|$$

$$\Longrightarrow \quad 3 = \left|(-3,-5,k-8) \cdot \frac{1}{7}(6,2,3)\right|$$

$$\Longrightarrow \quad 3 = \frac{1}{7}\left|(-3,-5,k-8) \cdot (6,2,3)\right|$$

$$\Longrightarrow \quad 21 = |-18 - 10 + 3k - 24|$$

$$\Longrightarrow \quad 21 = |3k - 52|$$

$$\Longrightarrow \quad 3k - 52 = \begin{cases} 21 \\ -21 \end{cases}$$

$$\Longrightarrow \quad 3k = \begin{cases} 73 \\ 31 \end{cases}$$

$$\Longrightarrow \quad k = \begin{cases} 73/3 \\ 31/3 \end{cases}$$

IYGB – FP1 PAPER V – QUESTION 7

USING STANDARD RESULTS & THE LINEARITY OF THE SIGMA OPERATOR

$$\Longrightarrow \sum_{r=1}^{n}(r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Longrightarrow \sum_{r=1}^{n}\left[r^2+(a+b)r+ab\right] \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Longrightarrow \sum_{r=1}^{n}r^2 + (a+b)\sum_{r=1}^{n}r + ab\sum_{r=1}^{n}1 \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Longrightarrow \frac{1}{6}n(n+1)(2n+1) + (a+b)\frac{1}{2}n(n+1) + ab\times n \equiv \frac{1}{3}n(n-1)(n+4)$$

$$\Longrightarrow n(n+1)(2n+1) + 3(a+b)n(n+1) + 6abn \equiv 2n(n-1)(n+4)$$

DIVIDING BY n, $n\neq 0$, AND EXPANDING BOTH SIDES

$$\Longrightarrow (n+1)(2n+1) + 3(a+b)(n+1) + 6ab \equiv 2(n-1)(n+4)$$

$$\Longrightarrow \cancel{2n^2}+3n+1 + 3(a+b)(n+1) + 6ab \equiv \cancel{2n^2}+6n-8$$

$$\Longrightarrow 3(a+b)(n+1) + 6ab \equiv 3n-9$$

$$\Longrightarrow 3(a+b)n + 3(a+b) + 6ab \equiv 3n-9$$

$\therefore\ 3(a+b)=3$

$\boxed{a+b=1}$

$3(a+b)+6ab=-9$

$a+b+2ab=-3$

$1+2ab=-3$

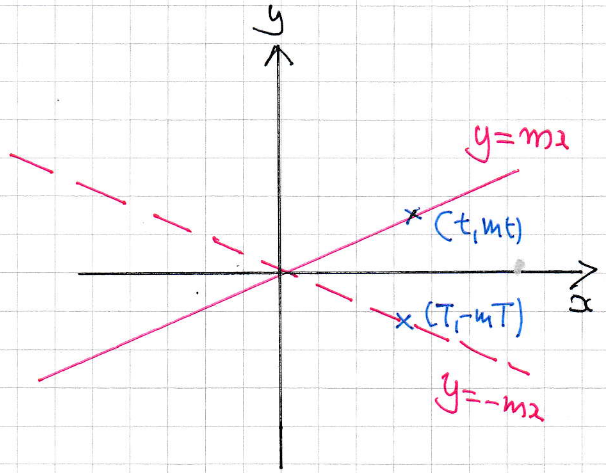
$2ab=-4$

$\boxed{ab=-2}$

& BY INSPECTION OR SOLVING WE OBTAIN $a=2$ & $b=-1$ IN ANY ORDER, AS EQUATIONS ARE SYMMETRIC

IYGB - FP1 PAPER V - QUESTION 8

LOOKING AT A DIAGRAM



under this transformation

$(t, mt) \longmapsto (T, -mT)$

HENCE WE OBTAIN

$$\begin{pmatrix} 1 & 2 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix} \Rightarrow \begin{pmatrix} t+2mt \\ 4t-7mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix}$$

$$\Rightarrow \left\{ \begin{array}{l} t+2mt = T \\ 4t-7mt = -mT \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} t(1+2m) = T \\ t(4-7m) = -mT \end{array} \right\}$$

$$\Rightarrow \frac{t(1+2m)}{t(4-7m)} = \frac{T}{-mT}$$

$$\Rightarrow \frac{1+2m}{4-7m} = \frac{1}{-m}$$

$$\Rightarrow -m-2m^2 = 4-7m$$

$$\Rightarrow 0 = 2m^2-6m+4$$

$$\Rightarrow m^2-3m+2=0$$

$$\Rightarrow (m-2)(m-1)=0$$

$$\Rightarrow m = \left\langle \begin{array}{l} 1 \\ 2 \end{array} \right.$$

IYGB - FP1 PAPER V - QUESTION 9

IF z IS REAL THEN $z = \bar{z}$

$$\Rightarrow z = (a+bi)^{4n} + (b+ai)^{4n}$$

$$\Rightarrow \bar{z} = \overline{(a+bi)^{4n} + (b+ai)^{4n}}$$

$$\Rightarrow \bar{z} = \overline{(a+bi)^{4n}} + \overline{(b+ai)^{4n}}$$

$\overline{z+w} \equiv \bar{z} + \bar{w}$

$$\Rightarrow \bar{z} = (\overline{a+bi})^{4n} + (\overline{b+ai})^{4n}$$

$\overline{z^n} \equiv \bar{z}^n$

$$\Rightarrow \bar{z} = (a-bi)^{4n} + (b-ai)^{4n}$$

$$\Rightarrow \bar{z} = [-i(b+ai)]^{4n} + [-i(a+bi)]^{4n}$$

$$\Rightarrow \bar{z} = (-i)^{4n}(b+ai)^{4n} + (-i)^{4n}(a+bi)^{4n}$$

$$\Rightarrow \bar{z} = [(-i)^4]^n(b+ai)^{4n} + [(-i)^4]^n(a+bi)^{4n}$$

$$\Rightarrow \bar{z} = 1^n(b+ai)^{4n} + 1^n(a+bi)^{4n}$$

$$\Rightarrow \bar{z} = (b+ai)^{4n} + (a+bi)^{4n}$$

$$\Rightarrow \bar{z} = (a+bi)^{4n} + (b+ai)^{4n}$$

$$\Rightarrow \bar{z} = z$$

INDEED REAL