FP1 – proof by mathematical induction Questions ANSWERS (73 marks)

Jan 2013

8.	(a) If $n=1$, $\sum_{r=1}^{n} r(r+3) = 1\times 4 = 4$ and $\frac{1}{3}n(n+1)(n+5) = \frac{1}{3}\times 1\times 2\times 6 = 4$,	B1
	(so true for $n = 1$. Assume true for $n = k$)	
	So $\sum_{r=1}^{k+1} r(r+3) = \frac{1}{3}k(k+1)(k+5)+(k+1)(k+4)$	M1
	$= \frac{1}{3}(k+1)\left[k(k+5)+3(k+4)\right] = \frac{1}{3}(k+1)\left[k^2+8k+12\right]$	A1
	$= \frac{1}{3}(k+1)(k+2)(k+6)$ which implies is true for $n = k+1$	dA1
	As result is true for $n=1$ this implies true for all positive integers and so result is true by induction	dM1A1cso
		(6)
	(b) $u_1 = 1^2(1-1)+1 = 1$	B1
	(so true for $n=1$. Assume true for $n = k$)	
	$u_{k+1} = k^2(k-1)+1+k(3k+1)$	M1,
	$= k(k^2-k+3k+1)+1 = k(k+1)^2+1$ which implies is true for $n = k+1$	A1
	As result is true for $n = 1$ this implies true for all positive integers and so result is true by induction	M1A1cso
		(5)
		[11]
Notes	(a) First B for LHS=4 and RHS =4	
	First M for attempt to use $\sum_{1}^{k} r(r+3)+u_{k+1}$	
	First A for $\frac{1}{3}(k+1)$, $\frac{1}{3}(k+2)$ or $\frac{1}{3}(k+6)$ as a factor before the final line	
	Second A dependent on first for $\frac{1}{3}(k+1)(k+2)(k+6)$ with no errors seen	
	Second M dependent on first M and for any 3 of 'true for n=1' 'assume true for n=k' 'true for n=k+1', 'true for all n' (or 'true for all positive integers') seen anywhere	
	Third A for correct solution only with all statements and no errors	

10.	$f(n) = 2^{2n-1} + 3^{2n-1}$ is divisible by 5.		
	$f(1) = 2^1 + 3^1 = 5$,	Shows that $f(1) = 5$.	B1
	Assume that for $n = k$, $f(k) = 2^{2k-1} + 3^{2k-1}$ is divisible by 5 for $k \in ¢^+$.		
	$f(k+1) - f(k) = 2^{2(k+1)-1} + 3^{2(k+1)-1} - \left(2^{2k-1} + 3^{2k-1}\right)$	M1: Attempts $f(k+1) - f(k)$. A1: Correct expression for <u>f(k + 1)</u> (Can be unsimplified)	M1A1
	$= 2^{2k+1} + 3^{2k+1} - 2^{2k-1} - 3^{2k-1}$		
	$= 2^{2k-1+2} + 3^{2k-1+2} - 2^{2k-1} - 3^{2k-1}$		
	$= 4\left(2^{2k-1}\right) + 9\left(3^{2k-1}\right) - 2^{2k-1} - 3^{2k-1}$	Achieves an expression in 2^{2k-1} and 3^{2k-1}	M1
	$= 3\left(2^{2k-1}\right) + 8\left(3^{2k-1}\right)$		
	$= 3\left(2^{2k-1}\right) + 3\left(3^{2k-1}\right) + 5\left(3^{2k-1}\right)$		
	$= 3f(k) + 5\left(3^{2k-1}\right)$		
	$\therefore f(k+1) = 4f(k) + 5\left(3^{2k-1}\right)$ or $4(2^{2k-1} + 3^{2k-1}) + 5\left(3^{2k-1}\right)$	Where $f(k+1)$ is correct and is clearly a multiple of 5.	A1
	If the result is true for $n = k$, then it is now true for $n = k+1$. As the result has shown to be true for $n = 1$, then the result is true for all n.	Correct conclusion **at the end**, at least as given, and all previous marks scored.	A1 cso
			[6]
			6 marks
	All methods should complete to f(k + 1) = ... where f(k +1) is clearly shown to be divisible by 5 to enable the final 2 marks to be available.		
Note that there are many different ways of proving this result by induction.			

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6(a)	$n=1, \text{LHS}=1^3=1, \text{RHS}=\frac{1}{4}\times1^2\times2^2=1$	Shows **both** LHS = 1 **and** RHS = 1	B1
	Assume true for n = k		
	When n = k + 1 $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2+(k+1)^3$	Adds $(k+1)^3$ to the given result	M1
	$=\frac{1}{4}(k+1)^2[k^2+4(k+1)]$	Attempt to factorise out $\frac{1}{4}(k+1)^2$	dM1
		Correct expression with $\frac{1}{4}(k+1)^2$ factorised out.	A1
	$=\frac{1}{4}(k+1)^2(k+2)^2$ Must see 4 things: true for n = 1, assumption true for n = k, said true for n = k + 1 and therefore true for all n	Fully complete proof with no errors and comment. All the previous marks must have been scored.	A1cso
	See extra notes for alternative approaches		(5)
(b)	$\sum(r^3-2)=\sum r^3-\sum 2$	Attempt two sums	M1
	$\sum r^3-\sum 2n$ is **M0**		
	$=\frac{1}{4}n^2(n+1)^2-2n$	Correct expression	A1
	$=\frac{n}{4}(n^3+2n^2+n-8)$ *	Completion to printed answer with no errors seen.	A1
			(3)
(c)	$\sum_{r=20}^{r=50}(r^3-2)=\frac{50}{4}\times130042-\frac{19}{4}\times7592$ $(=1625525-36062)$	Attempt $S_{50}-S_{20}$ or $S_{50}-S_{19}$ and substitutes into a correct expression at least once.	M1
		Correct numerical expression (unsimplified)	A1
	= 1 589 463	cao	A1
			(3)
(c) Way 2	$\sum_{r=20}^{r=50}(r^3-2)=\sum_{r=20}^{r=50}r^3-\sum_{r=20}^{r=50}(2)=\frac{50^2}{4}\times51^2-\frac{19^2}{4}\times20^2-2\times31$	M1 for ($S_{50}-S_{20}$ or $S_{50}-S_{19}$ for cubes) – (2x30 or 2x31)	**Total 11**
		A1 correct numerical expression	
	=1 589 463	A1	

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7(a)	$u_2 = 3,\ u_3 = 7$		B1, B1
			(2)
(b)	At $n = 1$, $u_1 = 2^1 - 1 = 1$ and so result true for $n = 1$		B1
	Assume true for $n = k$; $u_k = 2^k - 1$		
	and so $u_{k+1}\left(= 2u_k + 1\right) = 2(2^k - 1) + 1$	Substitutes u_k into u_{k+1} (**must see this line**)	M1
		Correct expression	A1
	$u_{k+1}\left(= 2^{k+1} - 2 + 1\right) = 2^{k+1} - 1$	Correct completion to $u_{k+1} = 2^{k+1} - 1$	A1
	Must see 4 things: <u>true for n = 1</u>, <u>assumption true for n = k</u>, <u>said true for n = k + 1</u> and therefore <u>true for all n</u>	Fully complete proof with no errors and comment. All the previous marks in (b) must have been scored.	A1cso
	Ignore any subsequent attempts e.g. $u_{k+2} = 2u_{k+1} + 1 = 2(2^{k+1} - 1) + 1$ **etc.**		(5)
			Total 7

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9. (a)	$n=1$; LHS $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ RHS $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ As LHS = RHS, the matrix result is true for $n = 1$.	Check to see that the result is true for $n = 1$.	B1
	Assume that the matrix equation is true for $n = k$, ie. $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$		
	With $n = k+1$ the matrix equation becomes $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$	$\begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$ by $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$	M1
	$= \begin{pmatrix} 3^{k+1}+0 & 0+0 \\ 9(3^k-1)+6 & 0+1 \end{pmatrix}$ or $\begin{pmatrix} 3^{k+1}+0 & 0+0 \\ 6.3^k+3(3^k-1) & 0+1 \end{pmatrix}$	Correct unsimplified matrix with no errors seen.	A1
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k)-3 & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3\left(3(3^k)-1\right) & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$	Manipulates so that $k \to k+1$ on at least one term.	dM1
		Correct result with no errors seen with some working between this and the previous A1	A1
	If the result is true for $n = k$, (1) then it is now true for $n = k+1$. (2) As the result has shown to be true for $n = 1$, (3) then the result is true for all n. (4) **All 4 aspects need to be mentioned at some point for the last A1.**	Correct conclusion with all previous marks earned	A1 cso

$f(1) = 7^{2-1} + 5 = 7 + 5 = 12$,	Shows that $f(1) = 12$.	B1
{which is divisible by 12}. $\{\therefore f(n)$ is divisible by 12 when $n = 1.\}$		
Assume that for $n = k$, $f(k) = 7^{2k-1} + 5$ is divisible by 12 for $k \in ¢^{+}$.		
So, $f(k+1) = 7^{2(k+1)-1} + 5$	Correct unsimplified expression for $f(k+1)$.	B1
giving, $f(k+1) = 7^{2k+1} + 5$		
$\therefore f(k+1) - f(k) = \left(7^{2k+1} + 5\right) - \left(7^{2k-1} + 5\right)$	Applies $f(k+1) - f(k)$. No simplification is necessary and condone missing brackets.	M1
$= 7^{2k+1} - 7^{2k-1}$		
$= 7^{2k-1}\left(7^2 - 1\right)$	Attempting to isolate 7^{2k-1}	M1
$= 48\left(7^{2k-1}\right)$	$48\left(7^{2k-1}\right)$	A1cso
$\therefore f(k+1) = f(k) + 48\left(7^{2k-1}\right)$, which is divisible by 12 as both $f(k)$ and $48\left(7^{2k-1}\right)$ are both divisible by 12.(1) If the result is true for $n = k$, (2) then it is now true for $n = k+1$. (3) As the result has shown to be true for $n = 1$,(4) then the result is true for all n. (5). **All 5 aspects need to be mentioned at some point for the last A1.**	Correct conclusion with no incorrect work. Don't condone missing brackets.	A1 cso
		(6)
There are other ways of proving this by induction. See appendix for 3 alternatives. **If you are in any doubt consult your team leader and/or use the review system.**		
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9.	$u_{n+1} = 4u_n + 2$, $u_1 = 2$ and $u_n = \frac{2}{3}(4^n - 1)$		
	$n = 1$; $u_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3}(3) = 2$ So u_n is true when $n = 1$.	Check that $u_n = \frac{2}{3}(4^n - 1)$ yields 2 when $n = 1$.	B1
	Assume that for $n = k$ that, $u_k = \frac{2}{3}(4^k - 1)$ is true for $k \in \mathbb{Z}^+$.		
	Then $u_{k+1} = 4u_k + 2$		
	$= 4\left(\frac{2}{3}(4^k - 1)\right) + 2$	Substituting $u_k = \frac{2}{3}(4^k - 1)$ into $u_{n+1} = 4u_n + 2$.	M1
	$= \frac{8}{3}(4)^k - \frac{8}{3} + 2$	An attempt to multiply out the brackets by 4 or $\frac{8}{3}$	M1
	$= \frac{2}{3}(4)(4)^k - \frac{2}{3}$		
	$= \frac{2}{3}4^{k+1} - \frac{2}{3}$		
	$= \frac{2}{3}(4^{k+1} - 1)$	$\frac{2}{3}(4^{k+1} - 1)$	A1
	Therefore, the general statement, $u_n = \frac{2}{3}(4^n - 1)$ is true when $n = k+1$. (As u_n is true for $n = 1$,) then u_n is true for all positive integers by mathematical induction	Require 'True when n=1', 'Assume true when n=k' and 'True when $n = k+1$' then true for all n o.e.	A1

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7.	(a) LHS = $f(k+1)=2^{k+1}+6^{k+1}$ $=2(2^k)+6(6^k)$ $=6(2^k+6^k)-4(2^k)=6f(k)-4(2^k)$	OR RHS = $=6f(k)-4(2^k)=6(2^k+6^k)-4(2^k)$ $=2(2^k)+6(6^k)$ $=2^{k+1}+6^{k+1}=f(k+1)$ (*)	M1 A1 A1 (3)
	OR $f(k+1)-6f(k)=2^{k+1}+6^{k+1}-6(2^k+6^k)$		M1
	$=(2-6)(2^k)=-4.2^k$, and so $f(k+1)=6f(k)-4(2^k)$		A1, A1 (3)
	(b) $n=1$: $f(1)=2^1+6^1=8$, **which is divisible by 8**		B1
	Either Assume f(*k*) divisible by 8 and try to use f(*k* + 1) = 6f(*k*) - $4(2^k)$ Show $4(2^k)=4\times 2(2^{k-1})=8(2^{k-1})$ or $8(\frac{1}{2}2^k)$ Or valid statement Deduction that result is implied for $n=k+1$ and so is true for positive integers by induction (may include *n* = 1 true here)	**Or** Assume f(*k*) divisible by 8 and try to use f (*k* + 1)– f (*k*) or f (*k* + 1) + f (*k*) including factorising $6^k=2^k3^k$ $=2^3 2^{k-3}(1+5.3^k)$ or $=2^3 2^{k-3}(3+7.3^k)$ o.e. Deduction that result is implied for $n=k+1$ and so is true for positive integers by induction (must include explanation of why *n* = 2 is also true here)	M1 A1 A1cso (4) **7 marks**
	Notes (a) M1: for substitution into LHS (or RHS) or $f(k+1)-6f(k)$ A1: for correct split of the two separate powers cao A1: for completion of proof with no error or ambiguity (needs (for example) to start with one side of equation and reach the other or show that each side separately is $2(2^k)+6(6^k)$ and **conclude** LHS = RHS) (b) B1: for substitution of *n* = 1 and stating "true for *n* = 1" or "divisible by 8" or tick. (This statement may appear in the concluding statement of the proof) M1: Assume f(*k*) divisible by 8 and consider f(*k* + 1) = 6f(*k*) - $4(2^k)$ or equivalent expression that could lead to proof – not merely f (*k*+1) – f(*k*) unless deduce that 2 is a factor of 6 (see right hand scheme above). A1: Indicates each term divisible by 8 **OR** takes out factor 8 or 2^3 A1: Induction statement . Statement *n* = 1 here could contribute to B1 mark earlier. NB: $f(k+1)-f(k)=2^{k+1}-2^k+6^{k+1}-6^k=2^k+5.6^k$ **only** is M0 A0 A0 (b) **"Otherwise" methods** Could use: $f(k+1)=2f(k)+4(6^k)$ or $f(k+2)=36f(k)-32(6^k)$ or $f(k+2)=4f(k)+32(2^k)$ in a similar way to given expression and Left hand mark scheme is applied. Special Case: Otherwise Proof **not involving induction**: This can only be awarded the B1 for checking *n* = 1.		

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Question	Scheme	Marks
9.	(a) If $n=1$, $\sum_{r=1}^{n} r^2 = 1$ and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}\times 1\times 2\times 3 = 1$, so true for $n = 1$. Assume result true for $n = k$ $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ $= \frac{1}{6}(k+1)(2k^2+7k+6)$ or $= \frac{1}{6}(k+2)(2k^2+5k+3)$ or $= \frac{1}{6}(2k+3)(k^2+3k+2)$ $= \frac{1}{6}(k+1)(k+2)(2k+3)$ $= \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1)$ or equivalent True for $n = k+1$ if true for $n = k$, (and true for $n = 1$) so true by induction for all n.	B1 M1 M1 A1 dM1 A1cso (6)
	Alternative for (a) After first three marks B M M1 as earlier : May state RHS $= \frac{1}{6}(k+1)(\{k+1\}+1)(2\{k+1\}+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$ for third M1 Expands to $\frac{1}{6}(k+1)(2k^2+7k+6)$ and show equal to $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ for A1 So true for $n = k+1$ if true for $n = k$, and true for $n = 1$, so true by induction for all n.	B1M1M1 dM1 A1 A1cso (6)
	(b) $\sum_{r=1}^{n}(r^2+5r+6) = \sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r +$ $\left(\sum_{r=1}^{n} 6\right)$ $\frac{1}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1)$, $+6n$ $= \frac{1}{6}n[(n+1)(2n+1) + 15(n+1) + 36]$ $= \frac{1}{6}n[2n^2+18n+52] = \frac{1}{3}n(n^2+9n+26)$ or $a = 9$, $b = 26$	M1 A1, B1 M1 A1 (5)
	(c) $\sum_{r=n+1}^{2n}(r+2)(r+3) = \frac{1}{3}2n(4n^2+18n+26) - \frac{1}{3}n(n^2+9n+26)$ $\frac{1}{3}n(8n^2+36n+52-n^2-9n-26) = \frac{1}{3}n(7n^2+27n+26)$ (*)	M1 A1ft A1cso (3) **14 marks**
	Notes: (a) B1: Checks $n = 1$ on both sides and states true for $n = 1$ here or in conclusion M1: Assumes true for $n = k$ (should use one of these two words) M1: Adds (k+1)th term to sum of k terms A1: Correct work to support proof M1: Deduces $\frac{1}{6}n(n+1)(2n+1)$ with $n = k+1$ A1: Makes induction statement. Statement true for $n = 1$ here could contribute to B1 mark earlier	