CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 3**

Model Solution No: 1

The general term in an arithmetic sequence is $u_n = a + (n-1)d$, where a is the first term and d is the common difference.

In order to find the sum of the first 100 terms, we first need to find a and d.

Since the 2nd term is 10, we know that $a + d = 10$

Since the 7th term is 28, we know that $a + 6d = 28$

If you solve these equations simultaneously, then you get $a = 32/5$ and $d = 18/5$

Hence the sum of the first 100 terms is

$$S_{100} = \frac{100}{2}\left[2\left(\frac{32}{5}\right) + (100-1)\left(\frac{18}{5}\right)\right] = 18460$$

which we have got by using the standard formula for the sum of the first n terms of an arithmetic sequence.

Answer: 18460

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Model Solution No: 2

(a) **Solution:** First we need to area element in terms of u.

$$u = \sqrt{x} \Rightarrow \mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x \Rightarrow \mathrm{d}x = 2u\,\mathrm{d}u$$

(Write $\sqrt{x}$ in index notation and then differentiate if you're not convinced.)

Now for the limits...

When $x = 0$, $u = \sqrt{0} = 0$.

When $x = \frac{\pi^2}{16}$, then $u = \sqrt{\frac{\pi^2}{16}} = \frac{\pi}{4}$

Thus using the substitution, we get:

$$\begin{aligned} I &= \int_0^{\frac{\pi^2}{16}} 4\sin(\sqrt{x})\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{4}} 4\sin(u)(2u\,\mathrm{d}u) \\ &= 8\int_0^{\frac{\pi}{4}} u\sin(u)\,\mathrm{d}u \end{aligned}$$

as required. (So $k = 8$, $a = 0$ and $b = \frac{\pi}{4}$.)

(b) We use integration by parts and apply the limits as we go. It is perfectly fine to find the indefinite integral and then apply the limits if you prefer.

$$
\begin{aligned}
\frac{I}{8} &= \int_0^{\frac{\pi}{4}} u \sin(u)\,\mathrm{d}u \\
&= \left[-u\cos(u)\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos u\,\mathrm{d}u \\
&= \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) + 0\right) + \left[\sin u\right]_0^{\frac{\pi}{4}} \\
&= -\frac{\pi\sqrt{2}}{8} + \sin\left(\frac{\pi}{4}\right) - \sin 0 \\
&= \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}
\end{aligned}
$$

Then multiplying everything by 8 gives the final answer

Answer: $I = 4\sqrt{2} - \pi\sqrt{2} = (4-\pi)\sqrt{2}$ (factorisation not required)

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Model Solution No: 3

(a) We have

$$\sqrt{16-2x} = \sqrt{16\left(1-\frac{x}{8}\right)} = 4\sqrt{1-\frac{x}{8}} = 4\left(1-\frac{x}{8}\right)^{\frac{1}{2}}$$

Hence using the binomial expansion we have

$$4\left(1-\frac{x}{8}\right)^{\frac{1}{2}} = 4\left(1+\frac{1}{2}\left(-\frac{x}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x}{8}\right)^2+o(x^3)\right)$$

which simplifies to $4-\frac{1}{4}x-\frac{1}{128}x^2$ for the first three terms

Answer: $4-\frac{1}{4}x-\frac{1}{128}x^2$

(b) First note that $f(4)=\sqrt{16-2(4)}=\sqrt{8}=2\sqrt{2}$

Then from our expansion we have

$$2\sqrt{2}\approx 4-\frac{1}{4}(4)-\frac{1}{128}(4)^2$$

If we work out the right hand side and divide we 2, we then get $\sqrt{2}\approx 1.4375$

Answer: 1.4375

(c) **Answer:** e.g. the binomial expansion of $f(x)$ is valid for $\left|\frac{x}{8}\right|<1$ which is equivalent to $|x|<8$. Hence it is valid to substitute $x=4$ into the binomial expansion of $f(x)$ because 4 is within this validity range.

Question Sheet: **Sheet 3**

Model Solution No: 4

(a) **Solution:**

$$\begin{aligned}\frac{\cos 2\theta}{\cos\theta+\sin\theta} &\equiv \frac{\cos^2\theta-\sin^2\theta}{\cos\theta+\sin\theta}\\ &\equiv \frac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{\cos\theta+\sin\theta}\\ &\equiv \cos\theta-\sin\theta\end{aligned}$$

as required.

(b) The equation is equivalent to

$$\frac{\cos 2\theta}{\sin\theta+\cos\theta}=\frac{1}{5}$$

which using part (a) is the same as $\cos\theta-\sin\theta=\frac{1}{5}$

To solve an equation like this, we need to express $\cos\theta-\sin\theta$ in the form $R\cos(\theta+\alpha)$ (for example).

For this expression, we require $R\cos\alpha=1$ and $R\sin\alpha=1$ (which you can check by writing out the expansion of $R\cos(\theta+\alpha)$ and comparing coefficients).

Thus, $R^2=1^2+1^2 \Rightarrow R=\sqrt{2}$ and $\tan\alpha=1\Rightarrow\alpha=\frac{\pi}{4}$. (These values have been obtained by (a) squaring both equations and adding and then (b) dividing the two equations.)

Hence $\cos\theta-\sin\theta=\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)$

Thus our equation to solve is

$$\cos\left(\theta+\frac{\pi}{4}\right)=\frac{1}{5\sqrt{2}}$$

Use your favourite method of solving trig equations to find that for our desired range $\theta=0.6$ and $\theta=4.1$ (1 dp).

Answer: $\theta=0.6$ and $\theta=4.1$

Question Sheet: **Sheet 3**

Model Solution No: 5

(a) **Solution:** The time to move from A to B is $\dfrac{30-x}{3} = \dfrac{30}{3} - \dfrac{1}{3}x = 10 - \dfrac{1}{3}x$ using time = speed/distance.

Now the distance between B and C is $\sqrt{8^2+x^2} = \sqrt{64+x^2}$ by Pythagoras. Hence the time taken to move from B to C is $\frac{\sqrt{64+x^2}}{2} = \frac{1}{2}\sqrt{64+x^2}$ again using time = distance/speed

Thus the total time to go from A to C is the combination of these two times:

$$T = 10 - \frac{1}{3}x + \frac{1}{2}\sqrt{64+x^2}$$

(b) We need to differentiate T wrt x. Now it is the third term that is the trickiest. However, we can write it in index form as $\dfrac{1}{2}(64+x^2)^{\frac{1}{2}}$. And now this is not too bad. We can differentiate this using the chain rule. The result is shown below - you can fill in the details as an exercise.

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x} &= -\frac{1}{3} + \frac{2x}{4\sqrt{64+x^2}} \\ &= -\frac{1}{3} + \frac{x}{2\sqrt{64+x^2}}\end{aligned}$$

Now T is minimised when $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$, so

$$\begin{aligned}&\frac{x}{2\sqrt{64+x^2}} = \frac{1}{3} \\ &\Rightarrow \sqrt{64+x^2} = \frac{3x}{2} \\ &\Rightarrow 64 + x^2 = \left(\frac{3x}{2}\right)^2 \\ &\Rightarrow 64 + x^2 = \frac{9x^2}{4} \\ &\Rightarrow \frac{5x^2}{4} = 64 \\ &\Rightarrow x^2 = \frac{256}{5} \\ &\Rightarrow x = \sqrt{\frac{256}{5}} = 7.155...\end{aligned}$$

Answer: Hence the value of x the minimises T is $x = 7.15$ (3 sf)

(ii) T is minimum at $x = 7.155...$, so to find the minimum value of T, we just plug this value of x into our expression for T. If you do this, you will find you get $T = 12.98...$

Answer: $T_{\text{min}} = 13.0$ s (3 sf)

(c) **Solution:** We need to find the second derivative and then substitute in $x = 7.155...$ and check the sign. To find the second derivative, we will need to use the quotient rule.

> Let's just write out all the derivatives before we rush straight to the formula. The quotient rule says that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$.
>
> In our case, $u = 2x$ and so $u' = 2$.
>
> We also have $v = 2\sqrt{64 + x^2} = 2(64 + x^2)^{\frac{1}{2}}$, so that (by the chain rule)
>
> $$v' = (64 + x^2)^{-\frac{1}{2}}(2x) = \frac{2x}{\sqrt{64 + x^2}}$$
>
> and we also have $v^2 = 4(64 + x^2)$
>
> Now we have all the information to find the second derivative:

$$\begin{aligned} \frac{\mathrm{d}^2T}{\mathrm{d}x^2} &= \frac{2\sqrt{64 + x^2}(2) - 2x\left(\frac{2x}{\sqrt{64+x^2}}\right)}{4(64 + x^2)} \\ &= \frac{32}{(64 + x^2)^{\frac{3}{2}}} \end{aligned}$$

The simplification was actually unnecessary though. What we care about is the sign of this second derivative at $x = 7.155...$. If you plug this value in to your second derivative, you get $T''(7.155...) = 0.0258... > 0$

Hence since $T''(7.155...) > 0$, this point does indeed minimise T.

(d) **Answer:** e.g. it assumes an instantaneous change of speed at the point B from 3 m s^{-1} to 2 m s^{-1} which is unrealistic / not valid for $x > 30$ (or $x < 0$)

Question Sheet: **Sheet 3**

Model Solution No: 6

(a) Resolve perpendicular to the plane using Newton's 2nd Law:

$$R - 2g\cos\theta = 0$$

because the resultant force is $R - 2g\cos\theta$ and the particle has 0 acceleration perpendicular to the plane. Hence we have that $R = 2g\cos\theta$

Now resolve parallel to the plane and in the direction of acceleration. Then

$$10 - \mu R - 2g\sin\theta = 0$$

On the left, we have a resultant force of $10 - \mu R - 2g\sin\theta$ because the 10 N force seeks to move the particle up the plane, while friction (μR) and gravity are opposing this upwards motion. It is still equal to 0 on the right because the particle is only on the point of moving up the plane, so the acceleration is still 0.

We want to find μ, the coefficient of friction, so re-arranging, we have

$$\mu = \frac{10 - 2g\sin\theta}{R} = \frac{10 - 2g\sin\theta}{2g\cos\theta}$$

and since $\tan\theta = \frac{5}{12}$, we can work this out. The answer is $\mu = 0.1360...$

Answer: $\mu = 0.136$

(b) Since P is a height of 4 m above the ground, its distance from the bottom of the slope is $\frac{4}{\sin\theta} = \frac{52}{5}$ (using a bit of right-angled trigonometry)

Now to find the time it takes to reach the bottom of the slope, we need to use N2L down the slope. Let's do this - resolving down the slope gives:

$$2g\sin\theta - \mu R = 2a$$

Notice now our forces on the left have changed. It is now gravity that is causing the motion down the slope and friction acts in the opposite direction to the motion. On the left hand side, we now have mass times acceleration because the particle is actually moving (it's not in 'equilibrium' or ' on the point of moving').

Using the values of μ and R from before, we get $a = 2.5384...$ m s^{-2}

Hence using SUVAT, the time taken to reach the bottom of the slope can be found (since the acceleration is uniform throughout the motion). You can fill in the details as an exercise, but as a hint, the equation necessary is $s = ut + \frac{1}{2}at^2$

Answer: $t = 2.86$ s (3 sf)

Question Sheet: **Sheet 3**

Model Solution No: 7

(a) **Answer:** Lowest pressure and windspeed should go to Beijing, so **A** is Beijing. Perth is close to the sea, so expect higher average windspeeds and pressures, so **C** is Perth. Location **B** is then Hurn.

(b) **Solution:** The hypotheses are $\mathrm{H}_0 : \rho = 0$, $\mathrm{H}_1 : \rho \neq 0$

This is a two-tailed test, so we are testing at the 5% level of significance in each tail (half the overall significance level). Then for a sample of size 100, the critical value for a significant correlation coefficient is $(-)0.1654$

For all three locations, the product moment correlation is smaller than -0.1654 (or larger in magntiude), so in all cases, reject H_0

There is evidence to suggest a (negative) correlation between p and w at the 10 % level of significance for all three locations.

(c) **Answer:** the product moment correlation coefficient for p and w is symmetric/the same for p against w as for w against p. Thus there is evidence to suggest a (negative) correlation between w and p at the 10 % level of significance for all three locations also.

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