CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

Question Sheet: **Sheet 8**

Model Solution No: 1

(a) This a disguised quadratic. Multiplying through by e^{2x} gives

$$3e^{4x} - 8 = 10e^{2x} \Rightarrow 3e^{4x} - 10e^{2x} - 8 = 0$$

This factorises to $(3e^{2x} + 2)(e^{2x} - 4) = 0$ (use the substitution $y = e^{2x}$ if it helps)

Now since e^{2x} must be positive (think about the graph), the only solution is when $e^{2x} = 4$. Taking logs gives $2x = \ln 4 \Rightarrow x = \frac{1}{2}\ln 4 = \ln 2$

Answer: $\ln 2$

(b)

$$\begin{aligned}
&\log_4(3-x) - 2\log_4(x) = 1\\
&\Rightarrow \log_4(3-x) - \log_4(x^2) = 1\\
&\Rightarrow \log_4\left(\frac{3-x}{x^2}\right) = 1\\
&\Rightarrow \frac{3-x}{x^2} = 4^1\\
&\Rightarrow 3 - x = 4x^2\\
&\Rightarrow 4x^2 + x - 3 = 0\\
&\Rightarrow (4x-3)(x+1) = 0
\end{aligned}$$

and since $0 < x < 3$, we have $x = \frac{3}{4}$.

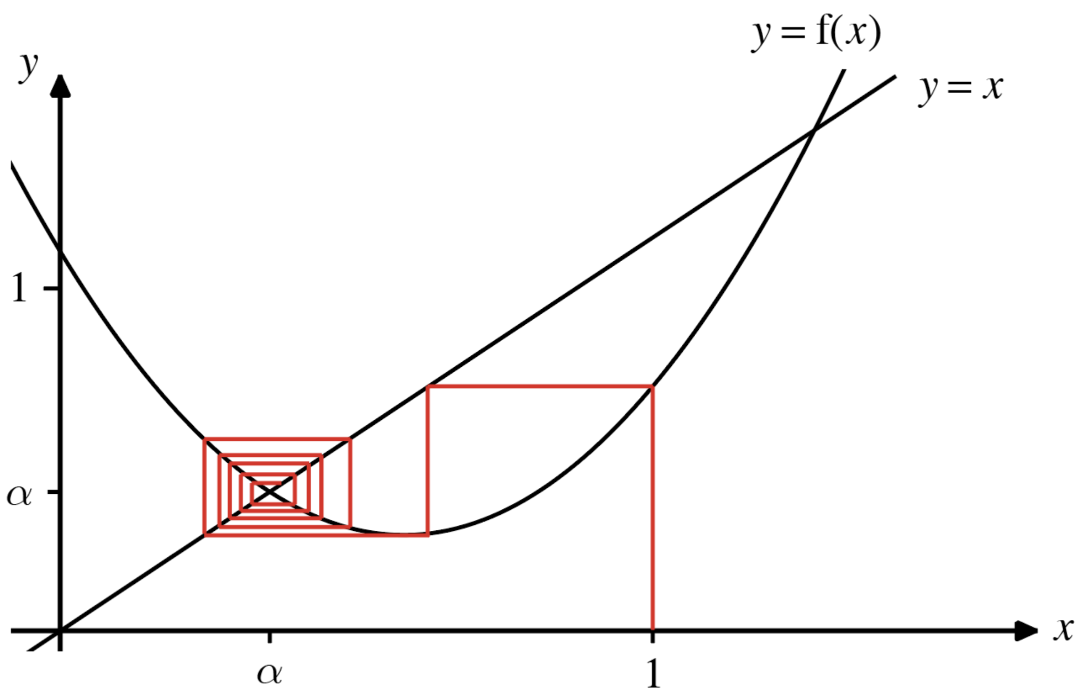
Answer: $x = \frac{3}{4}$

Extension: why do we need the restriction $0 < x < 3$?

Question Sheet: **Sheet 8**

Model Solution No: 2

Solution:



Yes, the iteration formula can be used to find α starting from $x_0 = 1$ because it converges to the root / we get a spiral inwards

For interest only: An interesting thing to notice is that no matter how close you are to the second root (the second point where the graphs intersect), provided you start to the left of it, the iteration formula will always converge to α - that seems weird because you expect if you start closer to the second root, the iteration would converge to that second root and not α. In this way, α is called a stable (fixed point) and the second root would be called a saddle (but of course this is terminology you don't need to know). But this demonstrates that the current iteration formula would not be suitable to find that second root starting from an initial value to the left of it.

Question Sheet: **Sheet 8**

Model Solution No: 3

(a) We have

$$\begin{pmatrix} 8 \\ p \\ -1 \end{pmatrix} = q \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + r \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

where we have taken the given information and written it in column vector notation. Now for this to be equal, all the components have to match. THis gives us the following three equations:

$$8 = 4q + 2r, \qquad p = q + 5r, \qquad -1 = -q - r$$

Now we can solve the first and last of these equations simultaneously for q and r to get $q = 3$ and $r = -2$. Then putting this into the second equation we find that $p = 3 + 5(-2) = -7$.

Answer: $p = -7$, $q = 3$, $r = -2$

(b) As with all vector questions, write out the path!! You want the position vector of D, so $\overrightarrow{OD}$. Now you think, how can I get from O to D using what we know. For example, I could go from O to A and then go from A to D or I could go from O to C and then from C to D. Let's try the first option:

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$

But there seems to be a problem: we don't know AD... *Or do we*? At some point, we're going to have to use the fact that D needs to make the shape a parallelogram and this is when. Since it is a parallelogram, we must have $\overrightarrow{AD} = \overrightarrow{BC}$ which is something we can work out. Now $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -12 \\ 0 \end{pmatrix}$ and thus

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \\ -1 \end{pmatrix}$$

Answer: $\overrightarrow{OD} = 10\mathbf{i} - 11\mathbf{j} - \mathbf{k}$

Extension: you can do part (b) using the path $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$. Do part (b) using this path and verify that you get the same answer.

Question Sheet: **Sheet 8**

Model Solution No: 4

(a) We need to use implicit differentiation. So differentiating both sides wrt x gives:

$$\begin{aligned}&2x + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0\\ &\Rightarrow 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x - 3y\\ &\Rightarrow (3x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -2x - 3y\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x-3y}{3x+2y}\end{aligned}$$

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x-3y}{3x+2y}$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+3y}{3x+2y}$

(b) If the tangent is parallel to the x axis, we must have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so thus $2x + 3y = 0$. This gives $y = -\frac{2}{3}x$.

Plugging this into the equation of the curve we get

$$\begin{aligned}&x^2 + 3x\left(-\frac{2}{3}x\right) + \left(-\frac{2}{3}x\right)^2 + 15 = 0\\ &\Rightarrow x^2 - 2x^2 + \frac{4}{9}x^2 + 15 = 0\\ &\Rightarrow -\frac{5}{9}x^2 + 15 = 0\\ &\Rightarrow x^2 = 27\\ &\Rightarrow x = \pm\sqrt{27} = x \pm 3\sqrt{3}\end{aligned}$$

Now $y = -\frac{2}{3}x$ and so when $x = 3\sqrt{3}$, $y = -2\sqrt{3}$ and when $x = -3\sqrt{3}$, $y = 2\sqrt{3}$

Answer: $(3\sqrt{3}, -2\sqrt{3})$, $(-3\sqrt{3}, 2\sqrt{3})$

Question Sheet: **Sheet 8**

Model Solution No: 5

(a) We seek A and B such that

$$\frac{1}{P(P-1)} = \frac{A}{P} + \frac{B}{P-1}$$

Multiplying through by $P(P-1)$, we have

$$1 = A(P-1) + BP$$

Now this holds for all P, so in particular let $P = 1$, then we find

$$1 = B$$

Likewise it holds for $P = 0$ also, which gives

$$1 = -A \Rightarrow A = -1$$

and thus $\dfrac{1}{P(P-1)} = -\dfrac{1}{P} + \dfrac{1}{P-1}$

Answer: $\dfrac{1}{P(P-1)} = -\dfrac{1}{P} + \dfrac{1}{P-1}$

(b) **Solution:** Separating variables, using part (a) and integrating gives

$$\begin{aligned}
\int \frac{1}{P(P-1)}\,\mathrm{d}P &= \int \frac{1}{4}\cos(3t)\,\mathrm{d}t \\
\Rightarrow \int \left(\frac{1}{P-1} - \frac{1}{P}\right)\,\mathrm{d}P &= \int \frac{1}{4}\cos(3t)\,\mathrm{d}t \\
\Rightarrow \ln(P-1) - \ln(P) &= \frac{1}{12}\sin(3t) + C
\end{aligned}$$

Using the initial condition $P = 4$ when $t = 0$, let's find the constant:

$$\ln(4-1) - \ln 4 = \frac{1}{12}\sin(0) + C \Rightarrow \ln 3 - \ln 4 = C$$

so $C = \ln\frac{3}{4}$ (if we combine the logs to make things neater). Thus we have

$$\ln(P-1) - \ln P = \frac{1}{12}\sin(3t) + \ln\frac{3}{4}$$

Now the goal is to get it in the required form. We're going to need to remove the logs, so let's combine the logs on the LHS side to get

$$\ln\left(\frac{P-1}{P}\right) = \frac{1}{12}\sin(3t) + \ln\frac{3}{4}$$

Now we can take exponentials of both sides at this stage (which works perfectly fine), but out of preference, let's first move the $\ln\frac{3}{4}$ on to the left hand side and combine it with the LHS, i.e.:

$$\ln\left(\frac{P-1}{P}\right) - \ln\frac{3}{4} = \frac{1}{12}\sin(3t)$$

which gives

$$\ln\left(\frac{4P-4}{3P}\right) = \frac{1}{12}\sin(3t)$$

Taking exponentials to both sides gives

$$\frac{4P-4}{3P} = \mathrm{e}^{\frac{1}{12}\sin(3t)}$$

Then re-arranging for P we have

$$\begin{aligned}
&4P - 4 = 3P\mathrm{e}^{\frac{1}{12}\sin(3t)} \\
&\Rightarrow 4P - 3P\mathrm{e}^{\frac{1}{12}\sin(3t)} = 4 \\
&\Rightarrow P(4 - 3\mathrm{e}^{\frac{1}{12}\sin(3t)}) = 4 \\
&\Rightarrow P = \frac{4}{4 - 3\mathrm{e}^{\frac{1}{12}\sin(3t)}}
\end{aligned}$$

as required.

(c) Remembering that P is measured in thousands, we want to solve

$$4.3 = \frac{4}{4 - 3\mathrm{e}^{\frac{1}{12}\sin(3t)}}$$

After some re-arrranging you get $\mathrm{e}^{\frac{1}{12}\sin(3t)} = \frac{44}{43}$. Taking logs gives

$$\frac{1}{12}\sin(3t) = \ln\frac{44}{43} \Rightarrow \sin(3t) = 12\ln\frac{44}{43}$$

Now we need to remember that calculus (as you have learnt it) only works in radians, so setting our calculator to radians we find that the first time the population reaches 4300 is

$$t = \frac{1}{3}\sin^{-1}\left(12\ln\frac{44}{43}\right) = 0.09316...$$

Answer: $t = 0.0932$ years (3 sf)

(c) The oscillation is caused by the $\sin(3t)$ term. $\sin(3t)$ oscillates between a maximum of 1 and a minimum -1. The maximum value of P will come from when the denominator is as small as possible (dividng by a smaller number gives a bigger result). Likewise the minimum value of P will come from when the denominator is as big as possible.

The denominator is its largest when $\sin(3t) = 1$ to give

$$P_{\text{min}} = \frac{4}{4 - 3\text{e}^{\frac{1}{12}(1)}} = 5.4106...$$

The denominator is smallest when $\sin(3t) = -1$ to give

$$P_{\text{max}} = \frac{4}{4 - 3\text{e}^{\frac{1}{12}(-11)}} = 3.226...$$

Answer: maximum value is 5.41 and minimum value is 3.23 (3 sf)

Question Sheet: **Sheet 8**

Model Solution No: 6

Start by resolving vertically. Using Newton's Second Law, we have

$$R - 30g - P\sin(30) = 0$$

because $R - 30g - P\sin(30)$ is the resultant vertical force and the RHS (mass times acceleration) is 0 since the particle is not moving vertically. This thus gives $R = 30g + \frac{1}{2}P$

Now resolve horizontally. Using Newton's Second Law again, we have

$$P\cos(30) - \mu R = 0$$

because $P\cos(30)$ is the horizontal force provided by the pushing and there is a frictional force in the direction opposite to this motion. Friction equals μR because we are moving so friction is at its maximum value. The RHS is 0 because the particle moves at constant speed, i.e. zero acceleration.

Plugging in the value of R found and $\mu = 0.3$, we get

$$\frac{\sqrt{3}}{2}P - 0.3\left(30g + \frac{1}{2}P\right) = 0$$

which if you re-arrange gives $P = 123.17...$ or $P = 120$ to 2 sf.

Answer: $P = 120$ (2 sf)

Question Sheet: **Sheet 8**

Model Solution No: 7

(a) **Answer:** 3.04, 2.95, 2.93, 2.83, 2.75

(b) (i) **Answer:** e.g. the product moment correlation coefficient measure the linear association between two variables

(ii) **Answer:** Answers which round to something in the range $[-0.981, -0.986]$

(iii) **Solution:** Hypotheses are: $\mathrm{H}_0 : \rho = 0$, $\mathrm{H}_1 : \rho \neq 0$.

We're doing a two-tailed test, so our significance level is 0.5 %. So the critical value for a sample of size 5 is $(-)0.9587$.

Since $0.986 > 0.9587$ (or $-0.986 < -0.9587$), reject H_0. There is evidence to suggest there is correlation between the data.

(c) Replacing x and y gives

$$\log P = -0.043t + 3.063$$
$$\Rightarrow P = 10^{-0.043t+3.063}$$
$$\Rightarrow P = 10^{3.063}(10^{-0.043})^t = 1160(0.906)^t$$

Answer: $P = 1160(0.906)^t$ (3 sf)

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