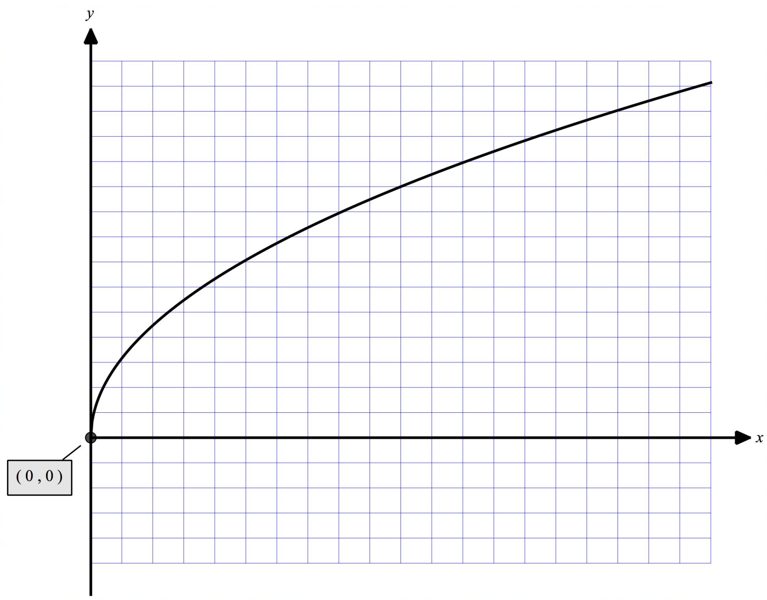
CRASHMATHS
SOLUTIONS TO QUESTION COUNTDOWN

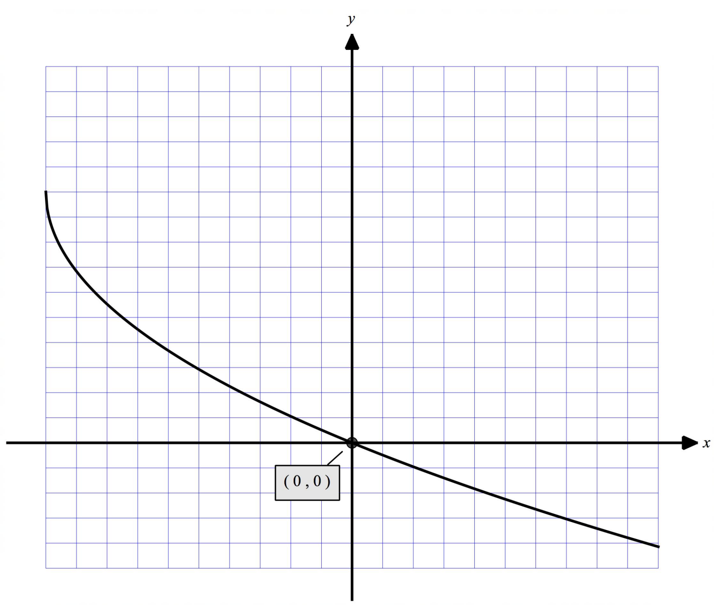
Question Sheet: **Sheet 4**

Model Solution No: 1

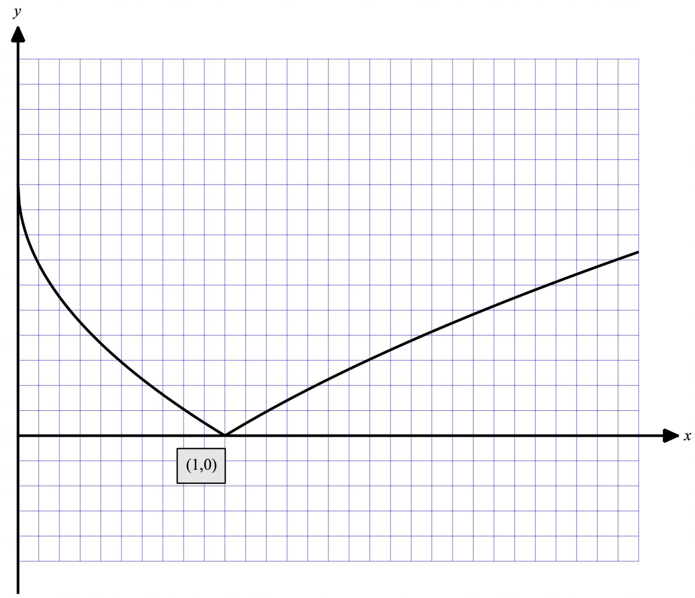
(a)



(b)



(c)



(d) **Answer:** the inverse of f exists since the function f is one-to-one

Question Sheet: **Sheet 4**

Model Solution No: 2

(a) Separating variables gives

$$e^{2y}\frac{dy}{dx} = 3x^2$$

Then integrate both sides with respect to x to get

$$\int e^{2y}\,dy = \int 3x^2\,dx$$

which then gives

$$\frac{1}{2}e^{2y} = x^3 + C$$

Substitute in the initial conditions to find the constant:

$$\frac{1}{2}e^{2(1)} = 0^3 + C \Rightarrow C = \frac{1}{2}e^2$$

Hence we now just have to get y in terms of x...

$$\begin{aligned}
&\Rightarrow \frac{1}{2}e^{2y} = x^3 + \frac{1}{2}e^2 \\
&\Rightarrow e^{2y} = 2x^3 + e^2 \\
&\Rightarrow 2y = \ln(2x^3 + e^2) \\
&\Rightarrow y = \frac{1}{2}\ln(2x^3 + e^2)
\end{aligned}$$

Answer: $y = \frac{1}{2}\ln(2x^3 + e^2)$ (allow 7.39 instead of e^2, but this is heavily discouraged - you should work with exact values where you can)

(b) Separating variables gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{1+x^2}$$

Then integrate both sides with respect to x to get

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

which then gives

$$\ln y = \frac{1}{2}\ln(1+x^2) + C$$

Substitute in the initial conditions to find the constant:

$$\ln 2 = \frac{1}{2} \ln 1 + C \Rightarrow C = \ln 2$$

Hence we now just have to get y in terms of x...

$$\begin{aligned}
&\Rightarrow \ln y = \frac{1}{2} \ln(1 + x^2) + \ln 2 \\
&\Rightarrow y = \mathrm{e}^{\frac{1}{2} \ln(1+x^2)+\ln 2} \\
&\Rightarrow y = \mathrm{e}^{\ln(\sqrt{1+x^2})} \times \mathrm{e}^{\ln 2} \\
&\Rightarrow y = 2\sqrt{1 + x^2}
\end{aligned}$$

Answer: $y = 2\sqrt{1 + x^2}$

(c) Note that $\mathrm{e}^{x-y} = \mathrm{e}^x \times \mathrm{e}^{-y}$. Hence separating variables gives

$$\mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^x$$

Then integrate both sides with respect to x to get

$$\int \mathrm{e}^y \, \mathrm{d}y = \int 3\mathrm{e}^x \, \mathrm{d}x$$

which then gives

$$\mathrm{e}^y = 3\mathrm{e}^x + C$$

Substitute in the initial conditions to find the constant:

$$\mathrm{e}^{\ln 3} = 3 + C \Rightarrow C = 3 - 3 = 0$$

Hence we now just have to get y in terms of x...

$$\begin{aligned}
&\Rightarrow \mathrm{e}^y = 3\mathrm{e}^x \\
&\Rightarrow y = \ln(3\mathrm{e}^x) \\
&\Rightarrow y = x + \ln 3
\end{aligned}$$

Answer: $y = x + \ln 3$

Question Sheet: **Sheet 4**

Model Solution No: 3

(a) **Solution:** Adding the two equations, we have $x + y = 2t$. Subtracting the two equations, we have $x - y = \frac{4}{t}$.

From the first, we have $t = \frac{x+y}{2}$. Putting this into the second gives

$$x - y = \frac{4}{\frac{x+y}{2}} \Rightarrow x - y = \frac{8}{x+y}$$

THen multiplying both sides by $(x + y)$ gives

$$(x - y)(x + y) = 8 \Rightarrow x^2 - y^2 = 8$$

as required with $P = 1$, $Q = -1$, $R = 8$

(b) By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

Now $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{2}{t^2} = \frac{t^2+2}{t^2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t^2-2}{t^2}$. Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2+2}{t^2} \times \frac{t^2}{t^2-2} = \frac{t^2+2}{t^2-2}$$

where we have used the fact that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\frac{\mathrm{d}t}{\mathrm{d}x}}$

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2+2}{t^2-2}$

As an alternative to (b), you can find the derivative by implicit differentiation on the Cartesian equation found in (a). Try this as an exercise and you should get the same derivative and stationary points.

(c) C has a stationary point if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. For this to happen, we need $t^2 + 2 = 0$ but this requires $t = \sqrt{-2}$ which doesn't admit real solutions. Hence C has no stationary points

Answer: $(2\sqrt{2}, 0)$

Question Sheet: **Sheet 4**

Model Solution No: 4

(a) From the geometry of the problem, the triangle OAB is isoceles. Call the length $OA = OB = r$, then $r^2 + r^2 = (3\sqrt{2})^2$ using Pythagoras. THis gives $r^2 = 9$ and thus $r = 3$.

So the coordinates of $A = (0, 3)$ and the coordinates of $B = (3, 0)$.

Answer: $A = (0, 3)$ and $B = (3, 0)$

(b) $M = (3, 3)$ because AM and BM must be radii of the circle since the coordinate axes are tangent. They intersect at $(3, 3)$ which has to be the centre because radii pass through the centre.

The radius of the circle is also 3.

Hence the equation of C is

$$(x - 3)^2 + (y - 3)^2 = 9$$

Answer: $(x - 3)^2 + (y - 3)^2 = 9$

(c) There are a few ways to do this. The easiest is to do the area of the square $OAMB$ - area of quarter of the circle.

The square $OAMB$ has area $3^2 = 9$ square units.

The quarter circle has area $\frac{1}{4}\pi \times 3^2 = \frac{9}{4}\pi$

Hence the area of the shaded region is $9 - \frac{9}{4}\pi = 1.931...$

Answer: 1.93 square units (3 sf)

Question Sheet: **Sheet 4**

Model Solution No: 5

(a) **Solution:** since the height of water is increasing with time, we require $\frac{\mathrm{d}H}{\mathrm{d}t} > 0$. Now $\frac{\mathrm{d}H}{\mathrm{d}t} = 0.2k\mathrm{e}^{0.2t}$ and this is only positive if $k > 0$. Hence k must be positive.

(b) Substitute $H = 1.5$ and $t = 2$ and re-arrange for k:

$$1.5 = k(\mathrm{e}^{0.4} - 1) \Rightarrow k = \frac{1.5}{\mathrm{e}^{0.4} - 1} = 3.049...$$

Answer: $k = 3.05$ (3 sf)

(c) Substitute $H = 4$ and re-arrange for t:

$$\begin{aligned} 5 &= (3.049...)(\mathrm{e}^{0.2t} - 1) \\ \Rightarrow \mathrm{e}^{0.2t} &= \frac{5}{3.049...} + 1 \\ \Rightarrow e^{0.2t} &= 2.6939... \\ \Rightarrow 0.2t &= \ln(2.6939...) \\ \Rightarrow t &= \frac{1}{0.2}\ln(2.6939...) = 4.852... \end{aligned}$$

Answer: $t = 4.85$ minutes

Tip: check your answer works by plugging $t = 4.85$ into the model for H. You should get an answer close to 5 (it won't be exactly 5 if you're using the unrounded version, but it should be very close).

Answer: e.g. decrease the value of the -1 (or increase its magnitude)

Question Sheet: **Sheet 4**

Model Solution No: 6

(a) Taking moments about A gives

$$3a(mg) + 4a(2mg) - 6a(T\sin 30) = 0$$

which gives $T = \frac{11mg}{3}$ after re-arranging.

Tip: Check dimensions. mg is a force and $\frac{11}{3}$ is a number with no dimensions. So our dimensions for T is a force.

(b) **Solution:** Resolving horizontally, we have $R - T\cos 30 = 0$ which gives

$$R = \frac{11}{3}mg\left(\frac{\sqrt{3}}{2}\right) = \frac{11\sqrt{3}}{6}mg$$

as required (where R denotes the horizontal component of the force from the wall on the rod. The suggestive letter R is deliberate - this force is indeed the normal reaction)

(c) Resolve vertically

$$mg + 2mg - T\sin 30 - F = 0$$

where F is the frictional force at the wall. Hence $F = \frac{7}{6}mg$

Now we are in equilibrium so $F \leq \mu R$. Note it is not = here because we are not in the case of limiting equilibrium, i.e. it is not on the point of sliding/moving. The maximum force of friction is μR, but we might not be at that limiting case so the frictional force may be a bit less than this...

Hence

$$\frac{7}{6}mg \leq \mu R$$

We know that $R = \frac{11\sqrt{3}}{6}mg$ and thus

$$\mu \geq \frac{7\sqrt{3}}{33}$$

Answer: $\mu \geq \frac{7\sqrt{3}}{33}$ (or awrt $\mu \geq 0.37$)

Question Sheet: **Sheet 4**

Model Solution No: 7

(a) The median is the 29.5th value. Hence using interpolation, we have (remember, your expression may look different to this):

$$\frac{10 - Q_2}{10 - 5} = \frac{38 - 29.5}{38 - 20}$$

which gives $Q_2 = 7.64$ after re-arranging to 3 sf

Answer: 7.64

(b) **Solution:** The lower quartile is given by

$$\frac{5 - Q_1}{5 - 0} = \frac{20 - 14.75}{20 - 0}$$

which gives the result of $Q_1 = 3.6875$ after re-arranging. This agrees with the required answer to 3 sf.

(c) As above, you can show the upper quartile is 11.4881. Hence the IQR is 11.4881 − 3.6875 = 7.8006

Answer: awrt 7.80

(d) **Answer:** To 3sf, mean is 7.58 and standard deviation is 4.17

(e) **Answer:** The median will increase. The lower quartile will increase (and the upper quartile will increase). The IQR will remain unchanged. The standard deviation will also not be affected. The mean will increase.

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