Worksheet 5 Solutions

Question 1 Solution.

(a) Since the curve crosses the y-axis at $(0, 1)$, this point must satisify the equation of the curve:

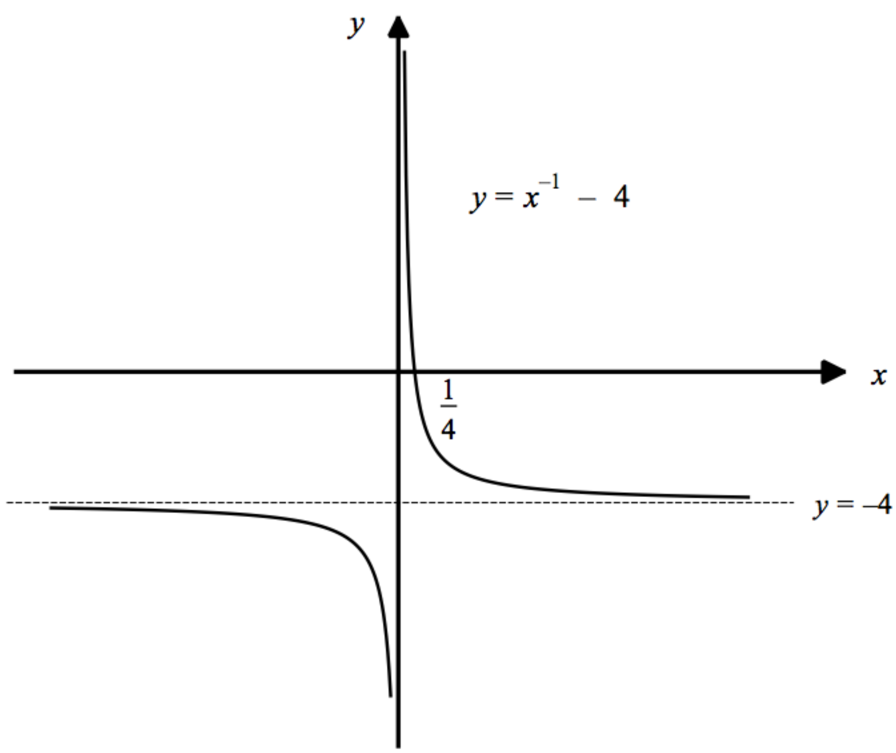
$$1 = \frac{2}{0+k} \Rightarrow 1 = \frac{2}{k} \Rightarrow k = 2$$

(b) The lines $y = 0$ and $x = -2$ are the asymptotes to the curve.

(c) $y = x^{-1} - 4 = \frac{1}{x} - 4$, which shifts the curve $y = \frac{1}{x}$ down by 4 units. Before we draw the sketch, let's note the key features:

- The curve cannot cross the y-axis, since x cannot be 0. So the y-axis is an asymptote.
- The curve crosses the x-axis when $y = 0$, so $0 = \frac{1}{x} - 4 \Rightarrow x = \frac{1}{4}$. So the coordinates of the point where the curve crosses the x-axis is $(\frac{1}{4}, 0)$.
- The line $y = -4$ is an asymptote to the curve

Now with that information, we can easily produce a sketch of the curve:



Question 2 Solution.

(a) Let's write y in index form, as usual: $y = x^3 - (px)^{\frac{1}{2}} = x^3 - p^{\frac{1}{2}}x^{\frac{1}{2}}$. So, remembering that p is just a constant, we have

$$\begin{aligned}\frac{dy}{dx} &= (3)x^2 - \frac{1}{2}p^{\frac{1}{2}}x^{-\frac{1}{2}} \\ &= 3x^2 - \frac{\sqrt{p}}{2\sqrt{x}}\end{aligned}$$

[You don't need to simplify the indices back to surds for the mark, if you write $\frac{dy}{dx} = 3x^2 - \frac{1}{2}p^{\frac{1}{2}}x^{-\frac{1}{2}}$, or equivalent, that is enough for the mark - this just looks nicer.]

(b) The gradient of the line $x + 2y + k = 0$ is $-\frac{1}{2}$.

So if the line is a normal to the curve at $x = 1$, then the curve has a gradient of 2 at $x = 1$. Therefore,

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=1} = 2 &\Rightarrow 3(1)^2 - \frac{\sqrt{p}}{2\sqrt{(1)}} = 2 \\ &\Rightarrow 6 - \sqrt{p} = 4 \\ &\Rightarrow \sqrt{p} = 2 \\ &\Rightarrow p = 4\end{aligned}$$

So the curve C has the equation $y = x^3 - 2\sqrt{x}$.

At $x = 1$, the y value of C is thus $y = 1 - 2(1) = -1$. Therefore the line must pass through the point $(1, -1)$. Substituing this into the line to find k, we get

$$1 + 2(-1) + k = 0 \Rightarrow k = 1$$

So we have $\boxed{p = 4}$ and $\boxed{k = 1}$.

Question 3 Solution.

The definition of the derivative of the curve $y = f(x)$ is

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If we look closely at the given expression, we can see that it corresponds to $f(x) = x^{-1}$. In other words,

$$\begin{aligned} \lim_{h \to 0} \left(\frac{(x+h)^{-1} - x^{-1}}{h} \right) &= \frac{d}{dx}\left(x^{-1}\right) \\ &= -x^{-2} \end{aligned}$$

So the answer is $\boxed{-x^{-2}}$.

Question 4 Solution.

We can set up two equations of motion for the particle and solve them simultaneously to find the values of a and b.

The particle does not move vertically and so it is in vertical equilibrium (i.e. its acceleration in this direction is 0). Applying Newton's 2nd Law ($F = ma$) and resolving upwards

$$R(\uparrow^{+}): \quad (2a+b) - 8 = 4(0) \Rightarrow 2a + b = 8 \tag{1}$$

Now we consider the horizontal motion of the particle. In this case it accelerates to the left at 8 m/s^2. So in this case, we resolve to the left (you can resolve to the right, but a will then be -8):

$$R({}^{+}\!\leftarrow): \quad a - (b-4) = 4(8) \Rightarrow a - b = 28 \tag{2}$$

We can solve equations (1) and (2) simultaneously by adding them to get: $3a = 36 \Rightarrow \boxed{a = 12}$ and so then $b = 12 - 28 = \boxed{-16}$

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