Complex Numbers

Solve Equations by Equating the Real and Imaginary parts

1) When you're given an equation involving complex numbers, you can **equate** the real parts and the imaginary parts **separately**. You can then form a pair of **simultaneous equations**.
2) You'll have to **substitute** $x + yi$ into an equation to represent a complex number — like in the examples below.

Example: Solve the equation $z^* + 5iz = 2 + 34i$.

Let $z = x + yi$, then $z^* = x - yi$.

$z^* + 5iz = x - yi + 5i(x + yi) = x - yi + 5xi + 5yi^2 = x - yi + 5xi - 5y = (x - 5y) + (5x - y)i$

$\Rightarrow (x - 5y) + (5x - y)i = 2 + 34i$

Equate the real parts and imaginary parts separately: ① $x - 5y = 2$ ② $5x - y = 34$

Solve these equations simultaneously: $5 \times$ ①: $5x - 25y = 10$

$5 \times$ ① − ②: $-24y = -24 \Rightarrow y = 1$

Plug this value for y into one of the equations: $x - 5(1) = 2 \Rightarrow x = 7$

Combine x and y back into a complex number: $z = x + yi$, so $z = 7 + i$

Two complex numbers are equal if and only if their real and imaginary parts are equal.

Example: Find the square roots of 8 + 6i.

Let $\sqrt{8+6i} = x + yi$, then $8 + 6i = (x + yi)^2 = x^2 + 2xyi + y^2i^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + 2xyi$

Equate the real and imaginary parts separately: ① $x^2 - y^2 = 8$ ② $2xy = 6 \Rightarrow y = \frac{3}{x}$

Solve simultaneously by substituting ② into ①: $x^2 - \frac{9}{x^2} = 8 \Rightarrow x^4 - 9 = 8x^2 \Rightarrow x^4 - 8x^2 - 9 = 0$

$(x^2 + 1)(x^2 - 9) = 0$

$x^2 = -1$ has no real solutions. (x and y must both be real numbers.)

$x^2 = 9 \Rightarrow x = 3$ and $x = -3$.

Plug these values for x back into the other equation: When $x = 3$, $y = 1$, and when $x = -3$, $y = -1$

So the roots are $3 + i$ and $-3 - i$.

Practice Questions

Q1 Write the real part, imaginary part and complex conjugate of each of these numbers:
a) $3 + 2i$ b) $-11 - 8i$ c) 5 d) $17i$

Q2 Write the following numbers as complex numbers in the form $a + bi$:
a) $(9 + 3i) - (5 - 2i)$ b) $(4 - 6i)(3i - 2)$ c) $\frac{3i+5}{2-9i}$

Q3 For the following complex numbers, write down the complex conjugate and compute zz^*:
a) $6 - 3i$ b) $-8 + 2i$ c) $14i$ d) 12

Exam Questions

Q1 Complex numbers z_1 and z_2 are given by $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$.
Giving your answers in the form $x + yi$, where x and y are real numbers, find:
a) $2z_1 - z_2$ [2 marks]
b) $(z_1z_1^*)(z_2z_2^*)$ [3 marks]
c) $\frac{z_1}{z_2}$ [3 marks]

Q2 Solve the following equations, giving your answers in the form $a + bi$:
a) $3iz^* + 4z = -1 - 6i$ [3 marks]
b) $4iz - 2z^* = 28 - 14i$ [3 marks]
c) $z^2 = i$ [4 marks]

$\sqrt{-1}$ love complex numbers...

After all these years, it turns out you actually can find the square root of negative numbers... The best thing is, the root of a complex number is also a complex number, so there's no need to invent any other types of numbers.

Complex Roots of Polynomials

Some Quartic Equations have no Real Roots

A **quartic equation** is a polynomial with an x^4 term (and nothing higher). So by the FTA it must have four roots, which can be: **4 real roots, 2 real roots and 1 pair of complex conjugate roots** or **2 pairs of complex conjugate roots.**

Example: A quartic equation is given by $f(x) = x^4 - x^3 - 3x^2 + 17x - 30$. Given that $f(-3) = f(2) = 0$, find the other two roots of $f(x)$.

1) $(x + 3)$ and $(x - 2)$ are factors of $f(x)$
$\Rightarrow (x + 3)(x - 2) = x^2 + x - 6$ is also a factor.

$$\begin{array}{r} x^2 - 2x + 5 \\ x^2 + x - 6\,\overline{)\,x^4 - x^3 - 3x^2 + 17x - 30} \\ -\underline{(x^4 + x^3 - 6x^2)} \\ -2x^3 + 3x^2 + 17x \\ +\underline{(2x^3 + 2x^2 - 12x)} \\ 5x^2 + 5x - 30 \\ -\underline{(5x^2 + 5x - 30)} \\ 0 \end{array}$$

This is an example of a quartic equation that has one pair of complex roots and 2 real roots

2) Find the other roots, where $x^2 - 2x + 5 = 0$, using the quadratic formula.

$$x = \frac{2 \pm \sqrt{(-2)^2 - (4 \times 1 \times 5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$
$$= 1 \pm 2i$$

So the other two roots are: $x = 1 + 2i$ and $x = 1 - 2i$

Example: Two roots of a quartic equation are $1 + 2i$ and $3 - 4i$. What is the quartic equation?

$1 + 2i$ is a root $\Rightarrow (1 + 2i)^* = 1 - 2i$ is a root
$3 - 4i$ is a root $\Rightarrow (3 - 4i)^* = 3 + 4i$ is a root
So the quartic equation is:

$$[x - (1 + 2i)][(x - (1 - 2i)][(x - (3 - 4i)][(x - (3 + 4i)]$$
$$= (x^2 - 2x + 5)(x^2 - 6x + 25)$$
$$= x^4 - 8x^3 + 42x^2 - 80x + 125$$

This is an example of a quartic equation with 2 pairs of complex roots and no real roots.

Steve was adamant that he had hair, and couldn't accept that the roots were imaginary

Practice Questions

In the following questions, all polynomials have real coefficients.

Q1 Give the missing roots of the following:
a) A quadratic equation where one of the roots is $1 - 7i$
b) A quartic equation where two of the roots are $2i$ and $5i - 2$

Q2 Find the quadratic function $f(x)$ that satisfies $f(1 + i) = 0$

Q3 By factorising the following polynomials, determine how many real roots and how many complex roots they each have:
a) $x^3 - x^2 + x - 1$ b) $x^4 - 16$ c) $x^2 + 1$ d) $x^4 - 2x^3 - x^2 + 2x$

Q4 Explain why a cubic equation can't have 3 complex roots, but a quartic equation can have 4 complex roots.

Exam Questions

Q1 The function $g(x)$ is given by $g(x) = x^4 - 2x^3 + 15x^2 - 134x + 290$.
Given that $3 - i$ is a root of $g(x)$, find the other three roots. [7 marks]

Q2 The function $f(x)$ is given by $f(x) = x^3 - 4x^2 + 17x - 26$.
a) Given that $f(x) = (x - 2)(x^2 + ax + b)$, find the values of a and b. [2 marks]
b) Hence, find the three roots of the equation $x^3 - 4x^2 + 17x - 26 = 0$. [3 marks]

Q3 $f(x) = x^3 + px^2 + qx - 20$, where p and q are real numbers.
Given that $f(2) = f(1 - 3i) = 0$, find the values of p and q. [3 marks]

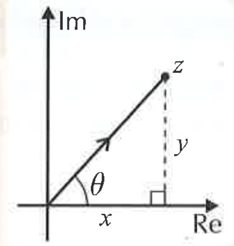
The exam is real and the numbers are imaginary — don't get it mixed up...

If in your pre-complex numbers life it upset you that there were "unsolvable" quadratics, I bet you're chuffed to bits now. And just because the solutions aren't real doesn't mean they're not useful — being able to find complex solutions to polynomials crops up in all sorts of places, such as in aerospace engineering and quantum mechanics.

Argand Diagrams

Every Number has a Modulus and an Argument

1) Recall that the **modulus** of a real number x, written $|x|$, is the **size** (or **length**) of the number — it's always a positive value.
2) This idea of the length of a number can be extended to the complex numbers. For $z = x + yi$, you can work out the modulus using **Pythagoras' theorem:** $|z| = \sqrt{x^2 + y^2}$.
3) Remember to only ever take the **positive square root.**
4) The **argument** of a complex number z, written **arg** z, is the angle between the **positive real axis** and the vector representing the complex number.
5) The argument is usually given in **radians**, where $-\pi < \arg z \leq \pi$. (This is sometimes called the **principal** argument.) So, measuring **anticlockwise** from the real axis gives a **positive** argument.
6) For $z = x + yi$, you can work out the angle, θ, between the vector and the real axis using $\theta = \tan^{-1}\left|\frac{y}{x}\right|$. You might have to do a bit more work to get the argument — see the example below:



$|z| = \sqrt{x^2 + y^2}$ $\tan\theta = \left|\frac{y}{x}\right|$

All positive real numbers have an argument of 0, and all negative real numbers have an argument of π.

Example: Find the modulus and principal argument of the following complex numbers to 2 decimal places:
a) $z_1 = 3 - 4i$ b) $z_2 = -4 + 2i$

a) $|z_1| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$\theta_1 = \tan^{-1}\left(\frac{4}{3}\right) = 0.9272...$ radians

So $\arg z_1 = -0.93$ **radians (2 d.p)**

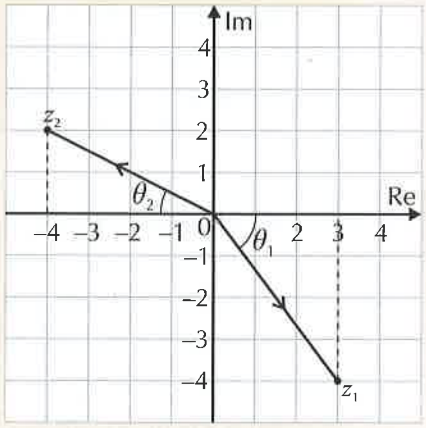
It's helpful to sketch the complex numbers on an Argand diagram.

b) $|z_2| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5} = 4.47$ **(2 d.p)**

$\theta_2 = \tan^{-1}\left(\frac{2}{4}\right) = 0.4636...$ radians

But θ_2 is the angle made with the negative real axis:

So $\arg z_2 = \pi - 0.4636...$ radians $= 2.68$ **radians (2 d.p)**



Practice Questions

Q1 Plot the complex numbers $s = 1 + 4i$ and $t = -4 - 5i$ on an Argand diagram and use these points to plot $t + s$.

Q2 Find the modulus for each of the following:
a) $5 - 6i$ b) $-4 + 4i$ c) $7i$ d) -9

Q3 Find the argument in radians to 2 d.p. for each of the following:
a) $2 - 6i$ b) $-9 + 3i$ c) 8 d) $-5i$

Exam Questions

Q1 Complex numbers s and t are given by $s = 2 - 5i$ and $t = 5 - 4i$.
a) Show s, t and $s - t$ on the same Argand diagram. [3 marks]
b) Find $|st|$. [3 marks]
c) Show that $\arg(s) + \arg(t) = \arg(st)$, giving any angles in radians. [3 marks]

Q2 The complex number z is given by $z = 8 - bi$, where b is a positive real number.
a) $|z| = 10$. Find the value of b. [1 mark]
b) Draw z and z^* on the same Argand diagram. [2 marks]
c) Use your answers to parts a) and b) to find $\arg(z)$ and $\arg(z^*)$ in radians, to 2 d.p. [2 marks]

Q3 Given that $-3 + i$ and $4 - 2i$ are roots of a quartic $f(x)$, find and plot all roots of $f(x)$ on an Argand diagram. [3 marks]

A grand diagram, don't you think...?

Argand diagrams are a fantastic way of seeing how the imaginary numbers fit with the real numbers. They're not just floating somewhere completely separate to the reals — they're right next to them, but going in a different direction...

Modulus-Argument Calculations

Divide Complex Numbers using the Modulus-Argument Form

1) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are complex numbers, then:

$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$. Multiplying top and bottom by $(\cos\theta_2 - i\sin\theta_2)$ gives:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} = \frac{r_1[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{r_2[\cos^2\theta_2 + \sin^2\theta_2]}$$

Then, by using the sin and cos addition formulas along with the identity $\cos^2\theta + \sin^2\theta \equiv 1$, we get:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

2) So **dividing** z_1 by z_2 gives a complex number with **modulus** $\frac{r_1}{r_2}$ and **argument** $\theta_1 - \theta_2$.

Example: Given that $p = 28[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)]$ and $q = 8(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})$, find $\frac{p}{q}$, giving your answer in the form $a + ib$, where a and b are given to 2 d.p.

$$\frac{p}{q} = \frac{28\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)}{8\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)} = \frac{28}{8}[\cos\left(-\frac{3\pi}{4} - \frac{2\pi}{5}\right) + i\sin\left(-\frac{3\pi}{4} - \frac{2\pi}{5}\right)]$$
$$= \frac{7}{2}[\cos\left(-\frac{23\pi}{20}\right) + i\sin\left(-\frac{23\pi}{20}\right)]$$
$$= -3.1185... + 1.5889...i = \mathbf{-3.12 + 1.59i\ (2\ d.p.)}$$

Summary:

If the modulus-argument forms of z_1 and z_2 are $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

$|z_1z_2| = |z_1||z_2| = r_1r_2$ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}$ $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$

You should try to learn these results off by heart.

Practice Questions

Q1 Write each of the following numbers in modulus-argument form giving angles in radians to 2 d.p.
a) $16 - 13i$ b) $-3 + 10i$ c) $-8 - 9i$

Q2 Find the product of $z_1 = 6(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and $z_2 = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.
Give your answer in the form $a + ib$.

Q3 Given that $p = 32[\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)]$ and $q = 12[\cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)]$,
find $\frac{p}{q}$, giving your answer in modulus-argument form.

Exam Questions

Q1 The complex number z is given by $z = -5 + 12i$. For the parts a), b) & c) give any angles in radians to 2 d.p.
a) Write z in modulus-argument form. [3 marks]
b) Find the value of $\frac{2z^*}{z^2}$. Give your answer in modulus-argument form. [4 marks]
c) Find the square roots of z. Give your answers in modulus-argument form. [7 marks]

Q2 Let $z_1 = 3 + 2i$ and $z_2 = -3 - 5i$.
a) Write z_1 and z_2 in modulus-argument form, giving any angles in radians to 2 d.p. [4 marks]
b) Compute $\frac{z_1}{z_2}$, giving your answer in the form $a + ib$, where a and b are given to 2 d.p. [3 marks]

No, the modulus is 4. Ok, let's not have a modulus-argument...

Finding the modulus of complex numbers is the easy bit — just use Pythagoras' theorem. Finding the argument is trickier — make sure you know which angle the tan formula is giving you, then use it to work out the argument. It all becomes much easier once you practise it — so if you haven't already, you know what to do...

Complex Loci

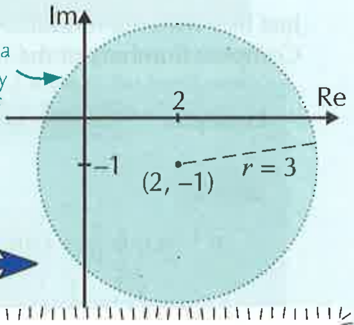
Complex Numbers can be used to describe Regions

1) If a complex locus is an **equation**, it describes a **line** or a **curve**.
But if the complex locus is an **inequality**, then it will describe a **region**.
2) Sometimes a region will be described by the set of points that satisfy **more than one** inequality. If it is, it could be written in set notation, so make sure you're familiar with what all the set symbols mean.

Example: a) Shade in the region represented by $|z - 2 + i| < 3$.

Firstly, solve the inequality as if it was an equation like we did on the previous page:
Set $z = x + yi$ and square both sides.
$|(x + yi) - 2 + i|^2 = 3^2 \Rightarrow (x - 2)^2 + (y + 1)^2 = 9$
The equation describes a circle centred at (2, –1), with radius 3.

You can use dotted lines to show points on a region boundary are not part of the shading.

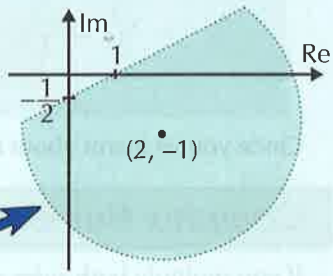


1) Now pick a point **inside** the circle (the centre is usually a good one), plug in the numbers and see if the inequality is satisfied.
$(2 - 2)^2 + (-1 + 1)^2 = 0 < 9$
2) Conclude which region you're looking for.
The point at the centre of the circle satisfies the inequality, so **the correct region is the inside of the circle.**

If the centre didn't satisfy the inequality, then the correct region would be everything outside the circle.

b) Hence, sketch the set of points $A = \{z \in \mathbb{C} : |z - 2 + i| < 3\} \cap \{z \in \mathbb{C} : |z + 2 - i| > |z + 3i|\}$.

1) Work out which points satisfy each subset, one at a time.
2) From part a), $|z - 2 + i| < 3$ means that the points will be inside the circle of radius 3 centred at (2, –1).
3) From the previous page, $|z + 2 - i| > |z + 3i|$ means the points must also satisfy $y < \frac{1}{2}(x - 1)$.
4) Pick a point to test, such as (0, 0). It does not satisfy $y < \frac{1}{2}(x - 1)$, which means the origin is not part of the shaded region.
5) So, shade the region that satisfies $y < \frac{1}{2}(x - 1)$ **and** is inside the circle of radius 3 centred at (2, –1).



Practice Questions

Q1 Describe and sketch the locus of z when: a) $|z| = 6$ b) $|z + 8| = |z - 4|$ c) $\arg(z + 2 + 2i) = \frac{2\pi}{3}$

Q2 On an Argand diagram, shade in the regions represented by:
a) $1 \leq |z + 2i| \leq 3$ b) $|z + 6| \geq |z + i|$
c) $0 \leq \arg(z + 2 + 3i) \leq \frac{\pi}{3}$, and find the Cartesian equations of the straight lines that bound the region.

Parts a) and c) have two inequalities here. Make sure your set of points satisfy both.

Exam Questions

Q1 The point P represents a complex number z on an Argand diagram such that $|z - 12i| = 2|z - 9|$
Find the equation of the locus of P, and describe its shape. [4 marks]

Q2 The point Q represents a complex number z on an Argand diagram such that $\arg(z - 5) = -\frac{3\pi}{4}$
a) Sketch, on an Argand diagram, the locus of Q as z varies. [2 marks]
b) Find **two** complex numbers for which both $|z + 4 + 3i| = 6$ and $\arg(z - 5) = -\frac{3\pi}{4}$ [5 marks]

Q3 Sketch the set of points $A = \{z \in \mathbb{C} : |z - 2i| > |z|\} \cap \{z \in \mathbb{C} : \frac{3\pi}{4} \leq \arg(z - 2) \leq \pi\}$ on an Argand diagram. [5 marks]

I'm still not sure how I'd describe the region where I live...

A lot of this stuff here should be fairly familiar — you'll have seen equations of circles and lines before, but you're just sticking them on an Argand diagram this time. That half-line business is a bit trickier though, so watch out for that one — if you're unsure which half of the line you need, just try a couple of different values to help you figure it out.

Roots of Polynomials

The Roots of a Quartic Equation are α, β, γ and δ

A **quartic** equation is a polynomial with an x^4 term in it (and no higher powers).
The **roots** of a quartic equation can also be linked to its **coefficients**:

If the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α, β, γ and δ then:

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a}, \quad \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a},$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}$$

If there are any repeated roots, replace the repeated roots with α as before.

You might see these written as $\sum\alpha$ (for $\alpha + \beta + \gamma + \delta$), $\sum\alpha\beta$ (for $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$) and $\sum\alpha\beta\gamma$ (for $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$). Remember that $\sum$ means **'sum of'**.

Example: A quartic equation has roots α, β, γ and δ, where $\sum\alpha = 4$, $\sum\alpha\beta = 5.5$, $\sum\alpha\beta\gamma = -8$ and $\alpha\beta\gamma\delta = 6$. Given that the coefficient of x^4 is 2, find the quartic equation.

$-\frac{b}{a} = 4$ and $a = 2$, so $b = -8$ $\quad$ $\frac{c}{a} = 5.5$ so $c = 11$ $\quad$ $-\frac{d}{a} = -8$ so $d = 16$ $\quad$ $\frac{e}{a} = 6$ so $e = 12$

So the equation is $\mathbf{2x^4 - 8x^3 + 11x^2 + 16x + 12 = 0}$.

The Roots might be Complex Numbers

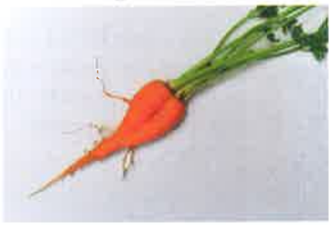
The relationships between the roots and the coefficients work for both real and complex roots (in fact, **all** the examples so far have had complex roots): Remember that if a complex number is a root, then its **complex conjugate** is also a root (see p.6-7).

Example: The equation $x^3 - 5x^2 + mx + n = 0$ has roots α, β and γ. If $\alpha = 1 + 2i$, what are the values of m and n?

Since $1 + 2i$ is a root, $1 - 2i$ is also a root, so $\boldsymbol{\beta = 1 - 2i}$.

In the equation, $a = 1$ and $b = -5$. So $\alpha + \beta + \gamma = 5 \Rightarrow \boldsymbol{\gamma = 3}$.

$m = \alpha\beta + \alpha\gamma + \beta\gamma = (1 + 2i)(1 - 2i) + 3(1 + 2i) + 3(1 - 2i) = \mathbf{11}$
and $n = -\alpha\beta\gamma = -3(1 + 2i)(1 - 2i) = \mathbf{-15}$



β was a bit complex, but α still liked him better than all the other roots.

Practice Questions

Q1 The equation $2x^3 + 8x^2 - 6x + 21 = 0$ has roots α, β and γ. Write down the values of:
a) $\alpha\beta\gamma$ $\quad$ b) $\alpha\beta + \alpha\gamma + \beta\gamma$ $\quad$ c) $\alpha + \beta + \gamma$

Q2 A cubic equation has roots α, β and γ, where $\alpha + \beta + \gamma = -\frac{5}{3}$, $\alpha\beta + \alpha\gamma + \beta\gamma = 7$ and $\alpha\beta\gamma = 10$.
Given that the coefficient of x^3 is 3, find the cubic equation.

Q3 The equation $(x + 3)(x + 5) = -7$ has roots α and β.
a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. $\quad$ b) Find the value of $\frac{\alpha^2 + \beta^2}{\alpha\beta}$.

Q4 The equation $x^4 - 6x^3 + 3x^2 + 7x - 9 = 0$ has roots α, β, γ and δ.
Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$.

Exam Questions

Q1 The equation $2x^3 + 4x^2 + 14x - 11 = 0$ has roots α, β and γ. Find the value of $(4 + \alpha)(4 + \beta)(4 + \gamma)$. [3 marks]

Q2 The equation $x^3 - gx^2 + h = 0$ has roots α, β and γ. Find the value of $\alpha^3 + \beta^3 + \gamma^3$ in terms of g and h. [4 marks]

Q3 The equation $x^4 - 12x^3 + 49x^2 + rx + 60 = 0$ has roots α, β, γ and δ.
If $\alpha = 2$ and $\beta = 6$, what is the value of r? [6 marks]

A polynomial's favourite road trip — Root 66...

To understand where all these formulas involving roots come from, have a go at deriving them for yourself. E.g. if α, β are the roots, then $a(x - \alpha)(x - \beta) = ax^2 + bx + c$ — so you can use this to derive the equations for the roots.

Related Roots

Method 2 — Use a Substitution to find the New Equation

You might find this alternative method easier to use for some polynomials:

1) Take the expression relating the **root** α of the original equation to the corresponding **new root** and **set it equal** to another letter, say y. Then rearrange to **express α in terms of y**.
2) **Substitute** the expression for α into the **original equation**.
3) **Rearrange** or **simplify** the new equation.

Example: The cubic equation $x^3 + 2x^2 - 5x + 1 = 0$ has roots α, β and γ. Find a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$.

1) Let $y = \alpha + 1$, then $\alpha = y - 1$.
2) Substitute $\alpha = y - 1$ into $x^3 + 2x^2 - 5x + 1$: $(y-1)^3 + 2(y-1)^2 - 5(y-1) + 1 = 0$.
   α is a root, so substituting the expression for α into the polynomial gives 0.
3) Expand the brackets and simplify: $y^3 - 3y^2 + 3y - 1 + 2(y^2 - 2y + 1) - 5y + 5 + 1 = 0$
   $\mathbf{y^3 - y^2 - 6y + 7 = 0}$
   It doesn't matter what variable you use for the equation, so it's fine to leave it in terms of y.

Example: The quartic equation $2x^4 + 5x^3 + 4x^2 - 7x + 11 = 0$ has roots α, β, γ and δ. Find a quartic equation with roots 3α, 3β, 3γ and 3δ.

1) Let $y = 3\alpha$, then $\alpha = \frac{y}{3}$.
2) Substitute $\alpha = \frac{y}{3}$ into $2x^4 + 5x^3 + 4x^2 - 7x + 11$: $2\left(\frac{y}{3}\right)^4 + 5\left(\frac{y}{3}\right)^3 + 4\left(\frac{y}{3}\right)^2 - 7\left(\frac{y}{3}\right) + 11 = 0$
3) Simplify the equation: $\frac{2y^4}{81} + \frac{5y^3}{27} + \frac{4y^2}{9} - \frac{7y}{3} + 11 = 0$
   Always give your answer with integer coefficients. → $\mathbf{2y^4 + 15y^3 + 36y^2 - 189y + 891 = 0}$

Practice Questions

All equations in your answers should be given with <u>integer coefficients</u>.

Q1 The quadratic equation $3x^2 + 4x + 9 = 0$ has roots α and β.
Use Method 1 to find the quadratic equation with roots 4α and 4β.

Q2 The cubic equation $x^3 + 6x^2 - 2x + 7 = 0$ has roots α, β and γ.
Use Method 1 to find the cubic equation with roots $\alpha + 2$, $\beta + 2$ and $\gamma + 2$.

Q3 The cubic equation $x^3 - 3x^2 + 7x - 4 = 0$ has roots α, β and γ.
Use Method 2 to find the cubic equation with roots $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$.

Q4 The quartic equation $x^4 + 9x^2 + 13 = 0$ has roots α, β, γ and δ.
Use Method 2 to find the quartic equation with roots $\alpha + 3$, $\beta + 3$, $\gamma + 3$ and $\delta + 3$.

Exam Questions

Q1 The cubic equation $3x^3 - 2x^2 - 4x - 6 = 0$ has roots α, β and γ.
Find the cubic equation with roots $3\alpha + 2$, $3\beta + 2$ and $3\gamma + 2$.
Your answer should have integer coefficients. [5 marks]

Q2 The quartic equation $5x^4 + 11x + 8 = 0$ has roots α, β, γ, and δ.
a) Use the substitution $x = 1 - 2u$ to obtain a quartic equation in terms of u. [2 marks]
b) Hence find the values of $\frac{1-\alpha}{2} + \frac{1-\beta}{2} + \frac{1-\gamma}{2} + \frac{1-\delta}{2}$ and $\left(\frac{1-\alpha}{2}\right)\left(\frac{1-\beta}{2}\right)\left(\frac{1-\gamma}{2}\right)\left(\frac{1-\delta}{2}\right)$. [3 marks]

He clearly dyes his equations — just look at those roots...

It's a good idea to learn both methods, but unless the question tells you what to do, it's up to you which method you use. As a bit of extra practice, try doing the examples on p.38 using Method 2, and those on p.39 using Method 1.

Summation of Series

The formula for the sum of r^3 can be proved in a similar way.

Example: Prove that when n is a positive integer, $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$.

When $n = 1$, LHS $= \sum_{r=1}^{1} r^3 = 1^3 = 1$

and RHS $= \frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4}(1)(2)^2 = 1$

So the statement is **true for $n = 1$**.

This follows exactly the same method as the previous example — show the statement's true for n = 1, then show that if it's true for n = k, it's also true for n = k + 1.

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

Then $\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$
$$= \frac{1}{4}(k+1)^2[k^2 + 4k + 4]$$
$$= \frac{1}{4}(k+1)^2(k+2)^2$$
$$= \frac{1}{4}(k+1)^2((k+1)+1)^2$$

We have shown that if the statement is true for $n = k$, then it is true for $n = k + 1$.
Since we have shown it to be true for $n = 1$, it must be true for all $n \geq 1$.

During his summation, Hamish began to wonder if the audience were entirely engaged.

Proof by induction isn't only useful for the standard summation results — you can use it to prove the summation formulas for many different series. Give these Practice and Exam Questions a go to see how it works for yourself.

Practice Questions

Q1 Use proof by induction to show that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, when n is any positive integer.

Q2 Use proof by induction to show that $\sum_{r=1}^{n} (r-1)(r+2) = \frac{1}{3}n(n+4)(n-1)$, when n is any positive integer.

Q3 Use proof by induction to show that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$, when n is any positive integer.

Q4 Use proof by induction to show that $\sum_{r=1}^{n} 3^r = \frac{3}{2}(3^n - 1)$, when n is any positive integer.

Exam Questions

Q1 Use proof by induction to show that $\sum_{r=1}^{n} r^2(r+1) = \frac{1}{12}n(n+1)(n+2)(3n+1)$, when n is any positive integer. [6 marks]

Q2 a) Use proof by induction to show that $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n+1)(n-4)$, when n is any positive integer. [6 marks]

b) For what value of n is $\sum_{r=1}^{n} r(r-3) = 10\sum_{r=1}^{n} r$ true? [3 marks]

Q3 a) Use proof by induction to show that $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$, when n is any positive integer. [6 marks]

b) For what value of n does $\sum_{r=1}^{n} r(r!)$ equal 23? [3 marks]

There are two more exciting episodes to come in this series...

The next page explains another method for showing a formula for the summation of a polynomial expression is correct — but if the question tells you to use proof by induction, make sure you use that method. When you're working out the sum for $n = k + 1$, remember what you're aiming for — that'll help you rearrange things to get the result you need.

Using Summations

Use the Method of Differences for Fractions

1) If an expression can be written as the **difference of similar terms**, then you can work out summations using the **method of differences**.
2) The method lets you **cancel** many of the **terms**, so that only the values **close to the upper and lower limits** of the sum actually **contribute** to the sum.
3) It's really handy for expressions that can be written as **partial fractions** — have a look at the example below to see how it works. It can also be used to prove some standard summation results — see Practice Question 3.

Example: Show that $\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$

First, rewrite the expression using partial fractions.

$$\frac{2}{(r+2)(r+4)} = \frac{A}{r+2} + \frac{B}{r+4} = \frac{A(r+4)+B(r+2)}{(r+2)(r+4)} = \frac{(A+B)r+(4A+2B)}{(r+2)(r+4)}$$

Equating coefficients: A + B = 0 and 4A + 2B = 2 ⇒ –2B = 2 ⇒ B = – 1 and A = 1.

$$\Rightarrow \frac{2}{(r+2)(r+4)} = \frac{1}{r+2} - \frac{1}{r+4}$$

You could also use the substitution method to find the partial fractions.

Write out the first few and last few terms in the sum.

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \sum_{r=1}^{n}\left[\frac{1}{r+2} - \frac{1}{r+4}\right]$$

$$= \left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\dots$$

$$\dots + \left(\frac{1}{(n-3)+2}-\frac{1}{(n-3)+4}\right)+\left(\frac{1}{(n-2)+2}-\frac{1}{(n-2)+4}\right)$$

$$+\left(\frac{1}{(n-1)+2}-\frac{1}{(n-1)+4}\right)+\left(\frac{1}{n+2}-\frac{1}{n+4}\right)$$

Look for terms that cancel — the only $\frac{1}{r+2}$ terms that won't be cancelled are $\frac{1}{3}$ and $\frac{1}{4}$. The only $\frac{1}{r+4}$ terms that won't be cancelled are $\frac{1}{n+3}$ and $\frac{1}{n+4}$.

$$= \frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\frac{1}{6}-\frac{1}{8}+\dots$$

$$\dots + \frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n}-\frac{1}{n+2}+\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+2}-\frac{1}{n+4}$$

$$= \frac{1}{3}+\frac{1}{4}-\frac{1}{n+3}-\frac{1}{n+4}$$

Put over a common denominator and simplify to get the required result.

$$= \frac{4(n+3)(n+4)+3(n+3)(n+4)-12(n+4)-12(n+3)}{12(n+3)(n+4)}$$

$$= \frac{7(n^2+7n+12)-24n-84}{12(n+3)(n+4)}$$

$$= \frac{7n^2+25n}{12(n+3)(n+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

Practice Questions

If you're doing AS Level, you only need to do Practice Question 1 and Exam Question 1.

Q1 Show that $\sum_{r=1}^{n} r^2(r-4) = \frac{n(n+1)}{12}(an^2+bn+c)$, where a, b and c are integers to be found.

Q2 a) Use the method of differences to show that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$. b) Find $\sum_{r=12}^{50} \frac{1}{r(r+1)}$.

Q3 Use the method of differences to prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. Hint: use $r = \frac{1}{2}(r(r+1) - r(r-1))$.

Exam Questions

Q1 Show that $4 + \sum_{r=1}^{n}(r+1)(r+4) = \frac{1}{3}(n+1)(n+2)(n+6)$. [6 marks]

Q2 a) Show that $\frac{2}{(r+2)(r+3)(r+4)}$ can be written in the form $\frac{A}{(r+2)(r+3)} + \frac{B}{(r+3)(r+4)}$, where A and B are constants to be found. [2 marks]

b) Use the method of differences to show that $\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)(r+4)} = \frac{n(n+7)}{24(n+3)(n+4)}$. [5 marks]

Summations and series — know your limits...

There were a few different ideas on these two pages, so make sure you're happy with all of them. A question might tell you which method to use when you're proving something involving series, so always read it carefully.

Volumes of Revolution

You can use *Parametric Equations* to find a *Volume*

Remember that curves can have **parametric equations** — where x and y are **functions of t**.
You can integrate these and you also need to be able to calculate their volumes of revolution.
Here are the **formulas** you need, for a curve with parametric equations $x = f(t)$ and $y = g(t)$ and limits t_1 and t_2.

For a rotation about the x-axis:

$$V = \pi \int_{t=t_1}^{t=t_2} y^2 \frac{dx}{dt}\, dt$$

Or for a rotation about the y-axis:

$$V = \pi \int_{t=t_1}^{t=t_2} x^2 \frac{dy}{dt}\, dt$$

Don't be put off if the equations use some other parameter instead of t (e.g. **trig** equations often use θ) — just change $\frac{dx}{dt}$ to $\frac{dx}{d\theta}$ and dt to $d\theta$ (or whatever the **parameter** is).

Example: A curve is given by the parametric equations $x = 2t$, $y = \sin 3t$. Find the volume formed when the area bounded by the curve, the x-axis and the lines $x = 0$ and $x = 2\pi$ is rotated 2π radians about the x-axis.

1) First of all, $\frac{dx}{dt} = 2$.

2) Change the limits:
$x = 0$, so $2t = 0 \Rightarrow t = 0$
$x = 2\pi$, so $2t = 2\pi \Rightarrow t = \pi$

3) Squaring y gives $y^2 = \sin^2 3t$.

Don't forget to change the limits from x to t.

4) Put the expressions for y^2, $\frac{dx}{dt}$ and the new limits into the formula:

$$V = \pi\int_0^\pi 2\sin^2 3t\, dt = \pi\int_0^\pi 1 - \cos 6t\, dt$$
$$= \pi\left[t - \frac{1}{6}\sin 6t\right]_0^\pi$$
$$= \pi\left[\pi - \frac{1}{6}\sin 6\pi\right] - \pi\left[0 - \frac{1}{6}\sin 0\right]$$
$$= \pi\left[\pi - \frac{1}{6}(0)\right] - 0 = \pi^2$$

This uses the identity $\cos 2t \equiv 1 - 2\sin^2 t$ — don't forget to double the coefficient of t.

Practice Questions

If you're doing AS Level, you only need to do Practice Question 1 and Exam Question 2.

Q1 What is the volume of the solid formed when the area enclosed by the curve $y = \sqrt{5 - 2x^2}$, the y-axis and the lines $y = 2$ and $y = 1$ is rotated 2π radians about the y-axis? Give an exact answer.

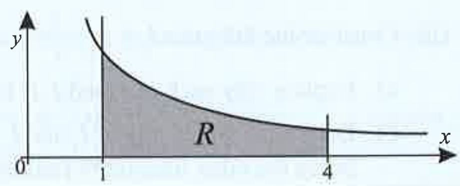
Q2 What sort of shape would you obtain by rotating the curve defined by the parametric equations $x = a\cos t$, $y = a\sin t$ by π radians around either of the axes? What would be its volume?

Q3 Find the volume of the solid obtained when the area enclosed by the y-axis, the lines $y = 1$ and $y = \pi + 1$ and the curve defined by the parametric equations $x = \cos 2\theta$, $y = 2\theta + 1$ is rotated 2π radians about the y-axis.

Exam Questions

Q1 A curve is defined by the parametric equations $x = 2t + 1$, $y = a\sec t$. Find the exact volume of the solid obtained when the area enclosed by this curve, the x-axis and the lines $x = 1 + \frac{\pi}{3}$ and $x = 1 + \frac{\pi}{2}$ is rotated 2π radians about the x-axis, leaving your answer in terms of a and simplifying as far as possible. [4 marks]

Q2 The graph below shows part of a curve with equation $y = \frac{\sqrt{2}}{(2x-1)^{\frac{1}{4}}}$.
The region R is the area enclosed by this curve, the x-axis and the lines $x = 1$ and $x = 4$.
A geography student is to make a scale model of a volcano using the solid obtained by rotating the region R a full revolution about the x-axis. The height of the real volcano, measured in the x-direction, is known to be 3 km.



a) Using the model, find the volume within the volcano. Give an exact answer. [3 marks]

b) Calculate the diameter, to 3 significant figures, of the volcano at:
i) its widest point, ii) its peak. [3 marks]

Come the revolution, I will have to kill you all...

Not to be confused with the French Revolution, the Industrial Revolution or the bloody CGP Revolution, volumes of revolution is just part of Further Maths A-Level. So don't go overthrowing your teachers and not letting them eat cake.