IYGB – FP1 PAPER Q – QUESTION 1

a) USING $|zw| = |z||w|$

$\Rightarrow |z_1 z_3| = 16$

$\Rightarrow |z_1||z_3| = 16$

$\Rightarrow |2-2i||z_3| = 16$

$\Rightarrow \sqrt{4+4}\,|z_3| = 16$

$\Rightarrow \sqrt{8}\,|z_3| = 16$

$\Rightarrow \sqrt{2}\sqrt{8}\,|z_3| = 16\sqrt{2}$

$\Rightarrow 4|z_3| = 16\sqrt{2}$

$\Rightarrow |z_3| = 4\sqrt{2}$

b) USING $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

$\Rightarrow \arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arg z_2 = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arg(\sqrt{3}+i) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \frac{\pi}{6} = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 = \frac{3\pi}{4}$

IYGB - FP1 PAPER Q - QUESTION 1

c) FINALLY IF $z_3 = a+bi$, $|z_3| = 4\sqrt{2}$, $\arg z_3 = \frac{3\pi}{4}$

$|a+bi| = 4\sqrt{2}$

$\sqrt{a^2+b^2} = 4\sqrt{2}$

$a^2+b^2 = 32$

$\arg z_3 = \frac{3\pi}{4}$

$\arctan\frac{b}{a} + \pi = \frac{3\pi}{4}$

$\arctan\frac{b}{a} = -\frac{\pi}{4}$

$\frac{b}{a} = \tan\left(-\frac{\pi}{4}\right)$

$\frac{b}{a} = -1$

$b = -a$

(OR SIMPLE INSPECTION)

$a^2+a^2 = 32$

$2a^2 = 32$

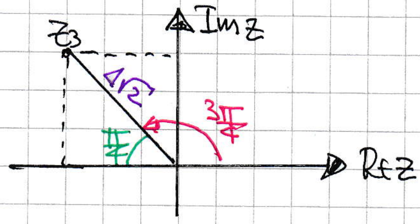
$a^2 = 16$

$a = -4$ (AS z_3 LIES IN THE 2ND QUADRANT)

& $b = +4$

FINALLY $\frac{z_3}{z_1} = \frac{-4+4i}{2-2i} = \frac{-2(2-2i)}{2-2i} = -2$ // AS REQUIRED

ALTERNATIVE FOR PART C



$z_3 = r(\cos\theta + i\sin\theta)$

$z_3 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_3 = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$z_3 = -4+4i$ ETC

IYGB - FP1 PAPER Q - QUESTION 2

OBTAIN RELATIONSHIPS FOR THE ROOTS OF THE GIVEN QUADRATIC

$$x^2+2x+3=0 \implies \begin{cases} \alpha+\beta = \frac{-2}{1} = -2 & \leftarrow -\frac{b}{a} \\ \alpha\beta = \frac{3}{1} = 3 & \leftarrow \frac{c}{a} \end{cases}$$

PROCEED AS FOLLOWS

$$\boxed{\begin{aligned} A &= \alpha - \frac{1}{\beta^2} \\ B &= \beta - \frac{1}{\alpha^2} \end{aligned}}$$

- $A+B = \left(\alpha-\frac{1}{\beta^2}\right)+\left(\beta-\frac{1}{\alpha^2}\right) = (\alpha+\beta)-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)$

$$= (\alpha+\beta) - \frac{\beta^2+\alpha^2}{\alpha^2\beta^2} = (\alpha+\beta) - \frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$

$$= (\alpha+\beta) - \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2} = -2 - \frac{(-2)^2-2\times3}{3^2} = -\frac{16}{9}$$

- $AB = \left(\alpha-\frac{1}{\beta^2}\right)\left(\beta-\frac{1}{\alpha^2}\right) = \alpha\beta - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\alpha^2\beta^2}$

$$= \alpha\beta - \left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + \frac{1}{(\alpha\beta)^2} = \alpha\beta - \left(\frac{\beta+\alpha}{\alpha\beta}\right) + \frac{1}{(\alpha\beta)^2}$$

$$= 3 - \frac{-2}{3} - \frac{1}{3^2} = 3 + \frac{2}{3} + \frac{1}{9} = \frac{34}{9}$$

HENCE THE REQUIRED EQUATION WILL BE

$$\implies x^2 - (A+B)x + AB = 0$$

$$\implies x^2 - \left(-\frac{16}{9}\right)x + \frac{34}{9} = 0$$

$$\implies 9x^2 + 16x + 34 = 0$$

IYGB - FP1 PAPER Q - QUESTION 3

$$\boxed{f(n) = 4^n + 6n - 1 \quad , \quad n \in \mathbb{N}}$$

BASE CASE

$f(1) = 4^1 + 6\times1 - 1 = 4+6-1 = 9$, IE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$ IE $f(k) = 9m$ WHERE $m \in \mathbb{N}$

$$\Rightarrow f(k+1) - f(k) = \left[4^{k+1} + 6(k+1) - 1\right] - \left[4^k + 6k - 1\right]$$

$$\Rightarrow f(k+1) - 9m = 4^{k+1} + \cancel{6k} + 6 \cancel{-1} - 4^k \cancel{-6k} \cancel{+1}$$

$$\Rightarrow f(k+1) - 9m = 4^{k+1} - 4^k + 6$$

$$\Rightarrow f(k+1) - 9m = 4\times4^k - 4^k + 6$$

$$\Rightarrow f(k+1) - 9m = 3\times4^k + 6$$

$f(k) = 4^k + 6k - 1$
$4^k = f(k) - 6k + 1$

$$\Rightarrow f(k+1) = 9m + 6 + 3\left[f(k) - 6k + 1\right]$$

$$\Rightarrow f(k+1) = 9m + 6 + 3f(k) - 18k + 3$$

$$\Rightarrow f(k+1) = 9m - 18k + 9 + 3(9m)$$

$$\Rightarrow f(k+1) = 36m - 18k + 9$$

$$\Rightarrow f(k+1) = 9\left[4m - 2k + 1\right]$$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

IYGB - FP1 PAPER Q - QUESTION 4

AS THE POLYNOMIAL EQUATION HAS REAL COEFFICIENTS ANY SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS, SO $z = 3 \pm i$ ARE SOLUTIONS

$$\begin{aligned}
[z-(3+i)][z-(3-i)] &= [(z-3)-i][(z-3)+i] \\
&= (z-3)^2 - i^2 \\
&= z^2 - 6z + 9 + 1 \\
&= z^2 - 6z + 10
\end{aligned}$$

BY LONG DIVISION

$$\begin{array}{r|l}
 & 2z^2 - 2z + 1 \\
\hline
z^2 - 6z + 10 & 2z^4 - 14z^3 + 33z^2 - 26z + 10 \\
 & -2z^4 + 12z^3 - 20z^2 \\
\hline
 & -2z^3 + 13z^2 - 26z + 10 \\
 & +2z^3 - 12z^2 + 20z \\
\hline
 & z^2 - 6z + 10 \\
 & -z^2 + 6z - 10 \\
\hline
\end{array}$$

HENCE $2z^2 - 2z + 1 = 0$

$$\begin{aligned}
4z^2 - 4z + 2 &= 0 \\
4z^2 - 4z + 1 &= -1 \\
(2z-1)^2 &= -1 \\
2z - 1 &= \pm i \\
2z &= 1 \pm i \\
z &= \tfrac{1}{2} \pm \tfrac{1}{2}i
\end{aligned}$$

$\therefore$ THE FULL SOLUTION SET IS $3+i,\ 3-i,\ \frac{1}{2}+\frac{1}{2}i,\ \frac{1}{2}-\frac{1}{2}i$

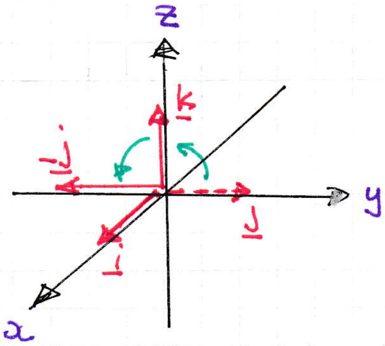
IYGB - FP1 PAPER Q - QUESTION 5

a) $$\boxed{\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}$$

$\underline{i} \mapsto \underline{i}$

$\underline{j} \mapsto \underline{k}$

$\underline{k} \mapsto -\underline{j}$



$\det \underline{A} = 1$ (NO REFLECTION)

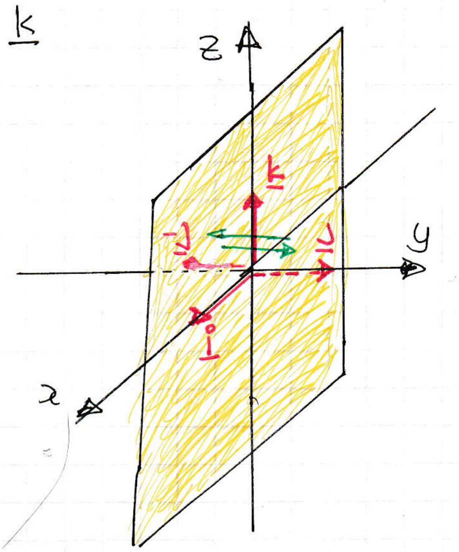
ROTATION ABOUT THE x AXIS, BY $90°$, ANTICLOCKWISE IN A RIGHT HAND SENSE

$$\boxed{\underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\underline{i} \mapsto \underline{i}$

$\underline{j} \mapsto -\underline{j}$

$\underline{k} \mapsto \underline{k}$



$\det \underline{B} = -1$ (REFLECTION INVOLVED)

REFLECTION ABOUT THE xz PLANE

b) COMPOSE IN THE CORRECT ORDER

$$\underline{C} = \underline{B}\,\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\det \underline{C} = -1$ (REFLECTION INVOLVED)

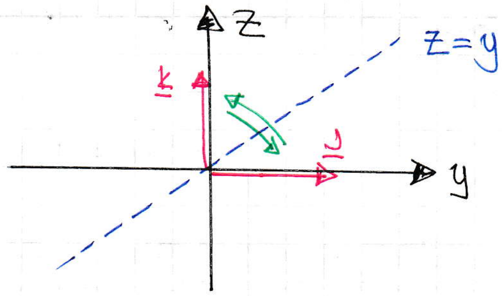
IYGB - FP1 PAPER Q - QUESTION 5

$\underline{i} \longmapsto \underline{i}$

$\underline{j} \longmapsto \underline{k}$

$\underline{k} \longmapsto \underline{j}$

LOOKING AT THE yz PLANE, FROM THE "POSITIVE" x



$\therefore$ REFLECTION ABOUT THE PLANE $y=z$

IYGB - FP1 PAPER Q - QUESTION 6

USING THE STANDARD SUMMATION FORMULAE

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)$$

HENCE WE NOW HAVE

$$\sum_{r=n}^{2n} (r^3-2r) = \sum_{r=n}^{2n} r^3 - 2\sum_{r=n}^{2n} r$$

$$= \left[\frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{4}(n-1)^2(n)^2\right] - 2\left[\frac{1}{2}(2n)(2n+1) - \frac{1}{2}(n-1)n\right]$$

$$= n^2(2n+1)^2 - \frac{1}{4}n^2(n-1)^2 - 2n(2n+1) + n(n-1)$$

$$= \frac{1}{4}n\left[4n(2n+1)^2 - n(n-1)^2 - 8(2n+1) + 4(n-1)\right]$$

AS IT WILL BE A MESS TO EXPAND TO FULL CUBIC IT BEST TO FACTORIZE THE TERMS INSIDE THE BRACKET IN PAIRS

$$= \frac{1}{4}n\left[n\left[4(2n+1)^2 - (n-1)^2\right] - 4\left[4n+2+1-n\right]\right]$$

$$= \frac{1}{4}n\left[n\underbrace{\left[4(2n+1)^2 - (n-1)^2\right]}_{\text{diff of squares}} - 4(3n+3)\right]$$

$$= \frac{1}{4}n\left[n\left[2(2n+1)-(n-1)\right]\left[2(2n+1)+(n-1)\right] - 12(n+1)\right]$$

$$= \frac{1}{4}n\left[n\left[4n+2-n+1\right]\left[4n+2+n-1\right] - 12(n+1)\right]$$

$$= \frac{1}{4}n\left[n(3n+3)(5n+1) - 12(n+1)\right]$$

$$= \frac{1}{4}n\left[3n(n+1)(5n+1) - 12(n+1)\right]$$

IYGB - FP1 PAPER Q - QUESTION 6

$$= \frac{3}{4}n\left[n(n+1)(5n+1) - 4(n+1)\right]$$

$$= \frac{3}{4}n(n+1)\left[n(5n+1) - 4\right]$$

$$= \frac{3}{4}n(n+1)(5n^2+n-4)$$

$$= \frac{3}{4}n(n+1)(5n-4)(n+1)$$

$$= \frac{3}{4}n(n+1)^2(5n-4)$$

ALTERNATIVE BY EXPANDING A CUBIC AFTER THE INITIAL FACTORING

$$\dots = \frac{1}{4}n\left[4n(2n+1)^2 - n(n-1)^2 - 8(2n+1) + 4(n-1)\right]$$

$$= \frac{1}{4}n\left[16n^3+16n^2+4n - n^3 + 2n^2 - n - 16n - 8 + 4n - 4\right]$$

$$= \frac{1}{4}n\left[15n^3+18n^2-9n-12\right]$$

$$= \frac{3}{4}n\left[5n^3+6n^2-3n-4\right]$$

LOOKING FOR FACTORS

$n=1 \qquad 5+6-3-4 \neq 0$

$n=-1 \qquad -5+6+3-4=0 \qquad \therefore (n+1)$ IS A FACTOR

LONG DIVIDE

$n+1$	$5n^3+6n^2-3n-4$
$5n^2+n-4$	$-5n^3-5n^2$
	n^2-3n-4
	$-n^2-n$
	$-4n-4$
	$4n+4$

$$\dots = \frac{3}{4}n(n+1)(5n^2+n-4)$$

$$= \frac{3}{4}n(n+1)(5n-4)(n+1)$$

$$= \frac{3}{4}n(n+1)^2(5n-4)$$

AS BEFORE

IYGB - FP1 PAPER Q - QUESTION 7

a) VOLUME OF REVOLUTION IN CARTESIAN, ABOUT THE x AXIS

$$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_2^6 \left(\frac{x+1}{\sqrt{x-1}}\right)^2 dx = \pi \int_2^6 \frac{(x+1)^2}{x-1} \, dx$$

BY SUBSTITUTION — OR — MANIPULATION

BY SUBSTITUTION

- $u = x-1$ OR $x = u+1$

$\frac{du}{dx} = 1$

$du = dx$

- $x=2 \mapsto u=1$

$x=6 \mapsto u=5$

$$\Rightarrow V = \pi \int_1^5 \frac{(x+1)^2}{u} \, du$$

$$\Rightarrow V = \pi \int_1^5 \frac{(u+1+1)^2}{u} \, du$$

$$\Rightarrow V = \pi \int_1^5 \frac{(u+2)^2}{u} \, du$$

$$\Rightarrow V = \pi \int_1^5 \frac{u^2+4u+4}{u} \, du$$

$$\Rightarrow V = \pi \int_1^5 u + 4 + \frac{4}{u} \, du$$

$$\Rightarrow V = \pi \left[\frac{1}{2}u^2 + 4u + 4\ln|u| \right]_1^5$$

$$\Rightarrow V = \pi \left[\left(\frac{25}{2} + 20 + 4\ln 5\right) - \left(\frac{1}{2} + 4 + \cancel{4\ln 1}\right) \right]$$

$$\Rightarrow V = \pi \left[28 + 4\ln 5 \right]$$

MANIPULATION

$$\Rightarrow V = \pi \int_2^6 \frac{[(x-1)+2]^2}{x-1} \, dx$$

$$\Rightarrow V = \pi \int_2^6 \frac{(x-1)^2 + 4(x-1) + 4}{x-1} \, dx$$

$$\Rightarrow V = \pi \int_2^6 \frac{(x-1)^2}{x-1} + \frac{4(x-1)}{x-1} + \frac{4}{x-1} \, dx$$

$$\Rightarrow V = \pi \int_2^6 x - 1 + 4 + \frac{4}{x-1} \, dx$$

$$\Rightarrow V = \pi \int_2^6 x + 3 + \frac{4}{x-1} \, dx$$

$$\Rightarrow V = \pi \left[\frac{1}{2}x^2 + 3x + 4\ln|x-1| \right]_2^6$$

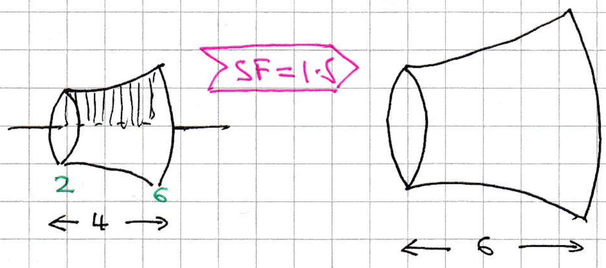
$$\Rightarrow V = \pi \left[(18 + 18 + 4\ln 5) - (2 + 6 + \cancel{\ln 1}) \right]$$

$$\Rightarrow V = \pi \left[28 + 4\ln 5 \right]$$

AS OPPOSITE

IYGB — FP1 PAPER Q — QUESTION 7

b) LOOKING AT THE SIMILAR SHAPES



$V = \pi(28 + 4\ln 5)$

$\mathbb{V} = ?$

$\mathbb{V} = V \times (\text{SCALE FACTOR})^3$

$\mathbb{V} = \pi(28 + 4\ln 5) \times (1.5)^3$

$\mathbb{V} \simeq 365$

IYGB – FP1 PAPER Φ – QUESTION 8

a) ELIMINATE TO CARTESIAN FIRST

$$\left.\begin{array}{l} x = 4+\lambda+5\mu \\ y = 8+2\lambda-4\mu \\ z = -5+\lambda+7\mu \end{array}\right\} \Longrightarrow \lambda = z+5-7\mu$$

SUBSTITUTE INTO THE FIRST TWO EQUATIONS

$$\left.\begin{array}{l} x = 4+(z+5-7\mu)+5\mu \\ y = 8+2(z+5-7\mu)-4\mu \end{array}\right\} \Rightarrow$$

$x = 9+z-2\mu$ $\quad \times 9$

$y = 18+2z-18\mu$ $\quad \times(-1)$

$$\left.\begin{array}{l} 9x = 81+9z-18\mu \\ -y = -18-2z+18\mu \end{array}\right\} \text{ADDING}$$

$9x-y = 63+7z$

$9x-y-7z = 63$

FINALLY THE EQUATION OF THE PLANE CAN BE WRITTEN AS

$(9,-1,-7)\cdot(x,y,z) = 63$

$$\mathbf{r}\cdot\begin{pmatrix}9\\-1\\-7\end{pmatrix} = 63$$

b) DETERMINE THE EQUATION OF A LINE THROUGH $P(12,-1,44)$ & IN THE DIRECTION OF THE NORMAL

$\mathbf{r} = (12,-1,44)+t(9,-1,-7)$

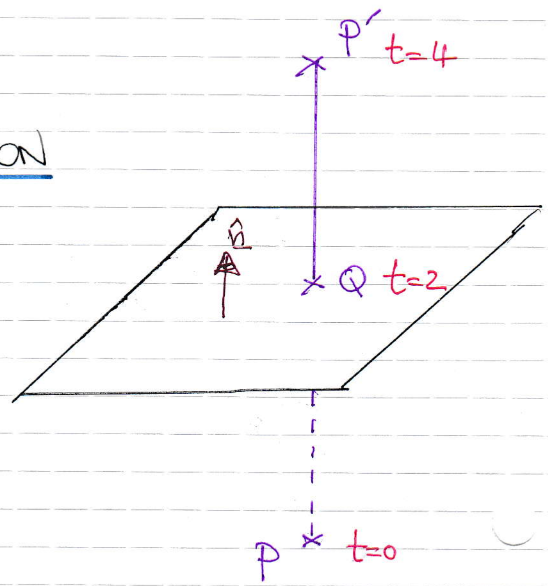
$(x,y,z) = (9t+12,\ -t-1,\ -7t+44)$

IYGB – FP1 PAPER Q – QUESTION 13

SOLVE SIMULTANEOUSLY WITH THE EQUATION OF THE PLANE

$x = 9t + 12$
$y = -t - 1$ & $9x - y - 7z = 63$
$z = 44 - 7t$

$\Rightarrow 9(9t+12) - (-t-1) - 7(44-7t) = 63$
$\Rightarrow 81t + t + 49t + 108 + 1 - 308 = 63$
$\Rightarrow 131t = 262$
$\Rightarrow t = 2$



USING $t=4$ WE OBTAIN THE REFLECTION

$P'(48, -5, 16)$

ALTERNATIVE/VARIATION

- USE $t=2$ TO FIND $Q(30, -3, 30)$

  THIS LIES ON
  $9x - y - 7z = 63$

- THEN USE "MIDPOINT PATTERNS"

$12 \xrightarrow{+18} 30 \xrightarrow{+18} 48$

$-1 \xrightarrow{-2} -3 \xrightarrow{-2} -5$

$44 \xrightarrow{-14} 30 \xrightarrow{-14} 16$

P Q P′