Worksheet 4 Solutions

Question 1 Solution.

First write $f(x)$ is index form so you can then differentiate: $f(x) = 2x^{-\frac{3}{2}} - x^{-3}$. Now, differentiating gives

$$\begin{aligned} f'(x) &= -\frac{3}{2}\left(2x^{-\frac{5}{2}}\right) + 3x^{-4} \\ &= -3x^{-\frac{5}{2}} + 3x^{-4} \end{aligned}$$

Now the maximum point will occur when $f'(x) = 0$, so

$$\begin{aligned} &-3x^{-\frac{5}{2}} + 3x^{-4} = 0 \\ &\Rightarrow x^{-\frac{5}{2}} = x^{-4} \\ &\Rightarrow x = 1 \end{aligned}$$

So the maximum value occurs when $x = 1$, and thus the maximum value is

$$f(1) = 2(1)^{-\frac{3}{2}} - 1^{-1} = 2 - 1 = \boxed{1}$$

Question 2 Solution.

(a) This is given by the value of N when $t = 0$:

$$N(0) = 120e^0 + e^0 = 121$$

so the number of individuals in the population before the outbreak is $\boxed{121}$.

(b)

$$\begin{aligned}\frac{dN}{dt} = 0 &\Rightarrow 2e^{2t} - 108e^{-0.9t} = 0 \\ &\Rightarrow e^{2t} = 54e^{-0.9t} \\ &\Rightarrow \ln(e^{2t}) = \ln(54e^{-0.9t}) \\ &\Rightarrow 2t = \ln(54) + \ln(e^{-0.9t}) \\ &\Rightarrow 2t = \ln(54) - 0.9t \\ &\Rightarrow 2.9t = \ln(54) \\ &\Rightarrow t = \frac{1}{2.9}\ln(54) = \frac{10}{29}\ln(54)\end{aligned}$$

which is an exact value as required. [NB: The final step of converting $\frac{1}{2.9} = \frac{10}{29}$ is good practice but not required.]

An alternative method to solving the equation in part (b) is to reach the stage $e^{2t} = 54e^{-0.9t}$, multiply both sides by $e^{0.9t}$ and then take logs.

Exam Tip: if a question asks for an exact value, do not resort to decimals - you will lose marks. You need to keep your answer in exact forms throughout and manipulate these exact forms with all the techniques you know.

(c) It decreases for $t < \frac{10}{29}\ln(54)$.

(d) The model predicts that $N \to \infty$ as $t \to \infty$, which is not realistic since the population cannot grow infinitely large as resources are limited.

Question 3 Solution.

(a) $\boxed{\text{True}}$.

Proof. Let p and q be rational numbers, then $p = \frac{a}{b}$ and $q = \frac{c}{d}$ for integers a, b, c and d (such that $b, d \neq 0$). Then

$$p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Clearly $ad + bc$ is an integer and bd is a non-zero integer, so $p + q$ is also rational.

(b) $\boxed{\text{True}}$.

Proof. Let p and q be rational numbers, then $p = \frac{a}{b}$ and $q = \frac{c}{d}$ for integers a, b, c and d (such that $b, d \neq 0$). Then

$$pq = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Clearly ac is an integer and bd is a non-zero integer, so pq is also rational.

(c) $\boxed{\text{False}}$.

Counterexample. For example, let $p = \sqrt{2}$ and $q = -\sqrt{2}$. Then

$$p + q = \sqrt{2} + (-\sqrt{2}) = 0$$

0 is not irrational, so the statement is false.

(d) $\boxed{\text{False}}$.

Counterexample. For example, let $p = \sqrt{2}$ and $q = \sqrt{2}$. Then

$$pq = (\sqrt{2}) \times (\sqrt{2}) = 2$$

2 is not irrational, so the statement is false.

Exam Tip: when a question asks for a counterexample, it is not enough to just give the counter-example. You need to state explicitly the counter-example and then show that the statement fails for these values.

Question 4 Solution.

(a) We have

$$\mathbf{F} = m\mathbf{a}$$

so

$$12\mathbf{i} - 8\mathbf{j} = 2\mathbf{a}$$

$$\Rightarrow \mathbf{a} = 6\mathbf{i} - 4\mathbf{j}$$

Therefore the magnitude of $\mathbf{a}$ is

$$|\mathbf{a}| = |6\mathbf{i} - 4\mathbf{j}| = \sqrt{6^2 + 4^2} = \boxed{2\sqrt{13}}$$

Now for the direction of $\mathbf{a}$

$$\theta = \tan^{-1}\left(\frac{4}{6}\right) = 33.69...$$

and so the direction as a bearing is $090 + 033.69... = \boxed{124}$ degrees (relative to north).

(b) Use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ to get

$$\mathbf{v} = (6\mathbf{i} + 4\mathbf{j}) + 3(6\mathbf{i} - 4\mathbf{j}) = 24\mathbf{i} - 8\mathbf{j}$$

This is the velocity of P after 3 seconds. The speed is the magnitude of this, so the speed after 3 seconds is $\sqrt{24^2 + 8^2} = \boxed{8\sqrt{10}}$.

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