# KS5 "Full Coverage": Integration (Year 2)

This worksheet is designed to cover one question of each type seen in past papers, for each A Level topic. This worksheet was automatically generated by the DrFrostMaths Homework Platform: students can practice this set of questions interactively by going to www.drfrostmaths.com, logging on, *Practise* → *Past Papers* (or *Library* → *Past Papers* for teachers), and using the 'Revision' tab.

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## Question 1

**Categorisation: Integrate exponential terms.**

*[OCR C3 June 2011 Q1i]* Find

$$\int 6e^{2x+1}dx$$

.......................... $+c$

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## Question 2

**Categorisation: In general, integrate expressions of the form $f'(ax+b)$**

Determine the following, using $c$ as your constant of integration.

$$\int \sin(3x)\,dx$$

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## Question 3

**Categorisation: Appreciate the need to expand and simplify to integrate some expressions.**

*[OCR C3 June 2016 Q2i]* Find

$$\int \left(2-\frac{1}{x}\right)^2 dx$$

.......................... $+c$

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## Question 4

**Categorisation: Split fractions in order to integrate.**

*[Edexcel A2 SAM P1 Q4]* Given that a is a positive constant and

$$\int_{a}^{2a} \frac{t+1}{t} dt = \ln 7$$

show that $a = \ln k$ , where $k$ is a constant to be found.

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## Question 5

**Categorisation: Integrate expressions of the form $(ax+b)^n$ where $n$ is negative or fractional.**

*[Edexcel C4 June 2016 Q7a]* Find

$$\int (2x-1)^{\frac{3}{2}} dx$$

giving your answer in its simplest form.

.......................... $+c$

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## Question 6

**Categorisation: Rewrite expressions in the form $\frac{1}{(ax+b)^n}$ as $(ax+b)^{-n}$ in order to integrate.**

Find

$$\int \frac{3}{\sqrt{5x-1}} \, dx$$

.......................... $+c$

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## Question 7

**Categorisation: Integrate common trig expressions.**

Determine $\int \ tan\ ^2 xdx$

........................ $+c$

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## Question 8

**Categorisation: As above.**

Determine $\int \ sec\ ^2 3xdx$

........................ $+c$

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## Question 9

**Categorisation: As above.**

Determine $\int \ \ cosec^2 xdx$

........................ $+c$

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## Question 10

**Categorisation: Integrate $sin^2 x$ and $cos^2 x$.**

Determine $\int \ sin\ ^2 xdx$

........................ $+c$

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## Question 11

**Categorisation: Integrate $a^x$ for some constant $a$.**

Determine $\int 3^x dx$

........................ $+c$

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## Question 12

**Categorisation: Integrate by inspection.**

Integrate the following, using $c$ as your constant of integration.

$$\int xe^{-x^2} dx$$

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## Question 13

**Categorisation: Integrate by inspection, appreciating that $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$**

Determine $\int \frac{x}{x^2+1} dx$

…………………….. $+c$

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## Question 14

**Categorisation: Integrate by splitting into partial fractions.**

*[Edexcel C4 June 2016 Q6i]*

Given that $y > 0$ , find

$$\int \frac{3y-4}{y(3y+2)} dy$$

…………………….. $+c$

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## Question 15

**Categorisation: Integrate by inspection for expressions of the form $\frac{1}{(ax+b)^n}$**

*[Edexcel C4 June 2014 Q6ii]* Find

$$\int \frac{8}{(2x-1)^3}\,dx$$

where $x > \frac{1}{2}$.

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; +c$

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## Question 16

**Categorisation: Appreciate the difference in techniques between integrating $\frac{1}{x}$ and integrating $\frac{1}{x^2}$**

*[Edexcel C4 June 2012 Q1bi Edited]*

$$f(x) = \frac{1}{x(3x-1)^2} = \frac{1}{x} - \frac{3}{3x-1} + \frac{3}{(3x-1)^2}$$

Find $\int f(x)dx$ .

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; +c$

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## Question 17

**Categorisation: Integrate by parts.**

*[Edexcel C4 June 2014 Q6i]* Find

$$\int xe^{4x}dx$$

$\int xe^{4x}dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; +c$

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## Question 18

**Categorisation: Definite integration by parts.**

*[Edexcel C4 Jan 2011 Q1]* Use integration to find the exact value of

$$\int_0^{\frac{\pi}{2}} x \sin 2x dx$$

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## Question 19

**Categorisation: Definite integration by parts to find an area.**

*[Edexcel A2 Specimen Papers P1 Q7]*

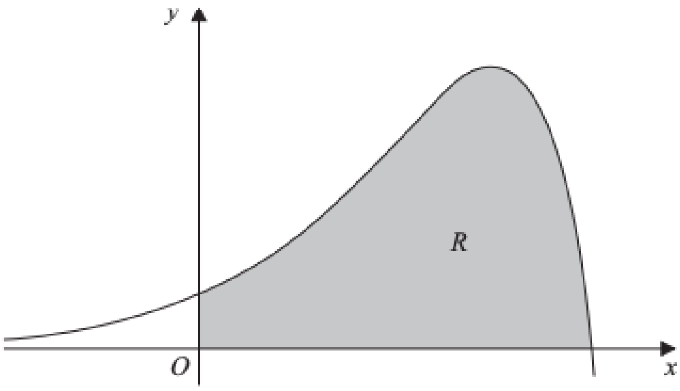


**Figure 4**

Figure 4 shows a sketch of part of the curve with equation $y = 2e^{2x} - xe^{2x}$ , $x \in$ $mathbb R$

The finite region $R$ , shown shaded in Figure 4, is bounded by the curve, the $x$ -axis and the $y$ -axis.

Use calculus to show that the exact area of $R$ can be written in the form $pe^4 + q$ , where $p$ and $q$ are rational constants to be found.

*(Solutions based entirely on graphical or numerical methods are not acceptable.)*

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## Question 20

**Categorisation: Integrate $\boldsymbol{ln\ x}$**

Integrate the following, using $c$ as your constant of integration.

$$\int ln\,(x)dx$$

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## Question 21

**Categorisation: Integrate by parts involving $\boldsymbol{ln\ x}$.**

*[Edexcel C4 June 2012 Q7b]*

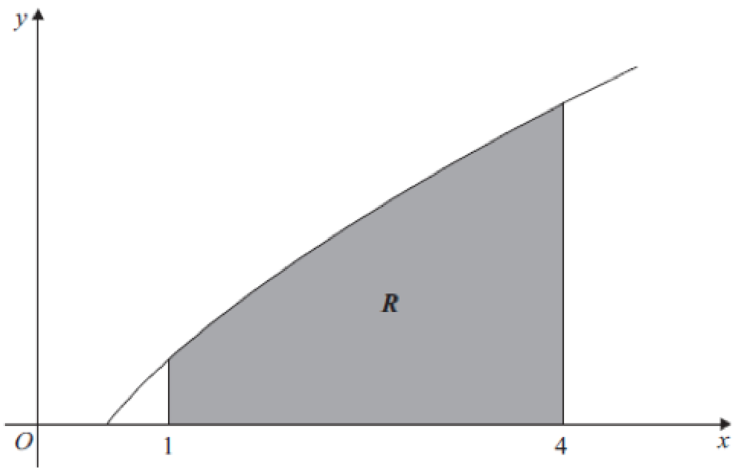


**Figure 3**

Figure 3 shows a sketch of part of the curve with equation $y = x^{\frac{1}{2}}\, ln\ 2x$ .

The finite region $R$, shown shaded in Figure 3, is bounded by the curve, the $x$ -axis and the lines $x = 1$ and $x = 4$ .

Find $\int x^{\frac{1}{2}}\, ln\ 2x dx$

…………………….. $+c$

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## Question 22
**Categorisation: As above.**

*[Edexcel A2 SAM P1 Q14c]*

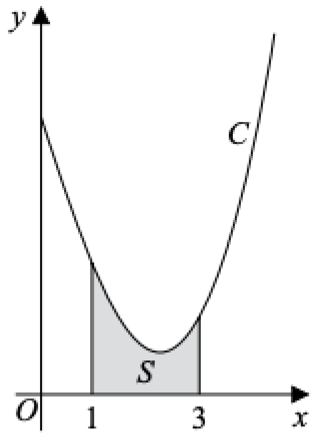


**Figure 4**

Figure 4 shows a sketch of part of the curve $C$ with equation

$y = \frac{x^2 \ln x}{3} - 2x + 5 , \quad x > 0$

The finite region $S$ , shown shaded in Figure 4, is bounded by the curve $C$ , the line with equation $x = 1$ , the $x$ -axis and the line with equation $x = 3$

Show that the exact area of $S$ can be written in the form $\frac{a}{b} + \ln c$ , where $a$ , $b$ and $c$ are integers to be found.

$S =$ ………………………

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## Question 23
**Categorisation: Integrate by parts twice.**

*[Edexcel C4 June 2013 Q1a]*

Find $\int x^2 e^x dx$

……………………… $+c$

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## Question 24
**Categorisation: Integrate by parts where the technique would infinitely loop.**

Determine $\int e^x \sin x dx$

.......................... $+c$

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## Question 25
**Categorisation: Integrate by substitution to determine an area.**

*[Edexcel C4 Jan 2013 Q4c]*

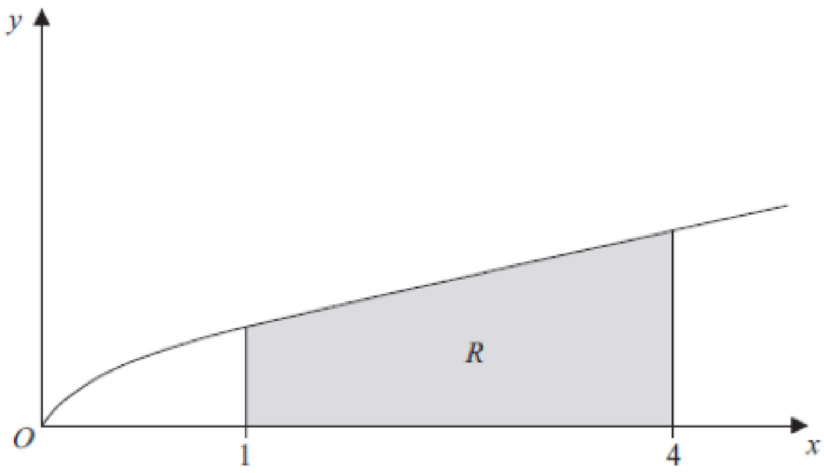


**Figure 1**

Figure 1 shows a sketch of part of the curve with equation $y = \frac{x}{1+\sqrt{x}}$ . The finite region *R*, shown shaded in Figure 1, is bounded by the curve, the $x$ -axis, the line with equation $x = 1$ and the line with equation $x = 4$ .

Use the substitution $u = 1 + \sqrt{x}$ to find, by integrating, the exact area of *R*.

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## Question 26

**Categorisation: Integrate by substitution.**

*[Edexcel C4 June 2010 Q2 Edited]*

Using the substitution $u = \cos x + 1$ , or otherwise, to find the exact value of

$$\int_0^{\frac{\pi}{2}} e^{\cos x+1} \sin x dx$$

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## Question 27

**Categorisation: Definite integration by substitution <u>when the substitution is not given.</u>**

*[Edexcel A2 Specimen Papers P1 Q12 Edited]*

Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta} d\theta = 2 + a \ln 2$$

where $a$ is a constant to be found.

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## Question 28
**Categorisation: As above.**

*[Edexcel C4 June 2016 Q6iia]*

Use the substitution $x = 4\,sin^2\theta$ to show that

$$\int_0^3 \sqrt{\frac{x}{4-x}}\,dx = \lambda \int_0^{\frac{\pi}{3}} sin^2\theta d\theta$$

where $\lambda$ is a constant to be determined.

$\lambda =$ ……………………..

## Question 29
**Categorisation: As above.**

*[Edexcel C4 Jan 2010 Q8a]*

Using the substitution $x = 2\,cos\,u$ , or otherwise, find the exact value of

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

……………………..

## Question 30

**Categorisation: As above.**

*[Edexcel C4 June 2013(R) Q3]* Using the substitution $u = 2 + \sqrt{2x+1}$ , or other suitable substitutions, find the exact value of

$$\int_0^4 \frac{1}{2+\sqrt{2x+1}}\,dx$$

giving your answer in the form $A + 2\ ln\ B$ , where $A$ is an integer and $B$ is a positive constant.

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## Question 31

**Categorisation: Parametric integration.**

*[Edexcel A2 Specimen Papers P2 Q10b Edited]*

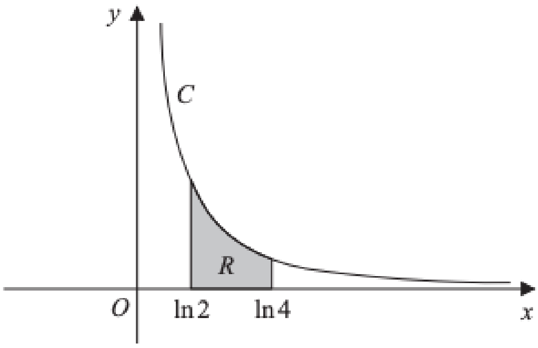


**Figure 4**

Figure 4 shows a sketch of the curve $C$ with parametric equations

$$x = \ ln\ (t+2), \qquad y = \frac{1}{t+1}, \qquad t > -\frac{2}{3}$$

The finite region $R$ , shown shaded in Figure 4, is bounded by the curve $C$ , the line with equation $x = \ ln\ 2$ , the $x$ -axis and the line with equation $x = \ ln\ 4$

Use calculus to find the exact area of $R$ .

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## Question 32

**Categorisation: As above.**

*[Edexcel C4 Jan 2010 Q7b Edited]*

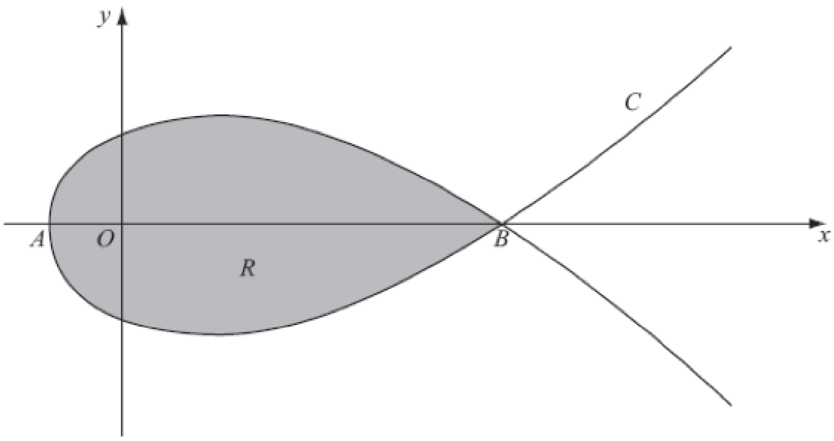


**Figure 2**

$x = 5t^2 - 4\,, \quad y = t(9 - t^2)$

The curve *C* cuts the $x$-axis at the points $A$ and $B$. The $x$-coordinate at the point $A$ is $-4$ and the $x$-coordinate at the point $B$ is $41$.

The region *R*, as shown shaded in Figure 2, is enclosed by the loop of the curve.

Use integration to find the area of *R*.

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# Question 33

**Categorisation: Parametric integration involving an $a^x$ term.**

*[Edexcel C4 Jan 2013 Q5d Edited]*

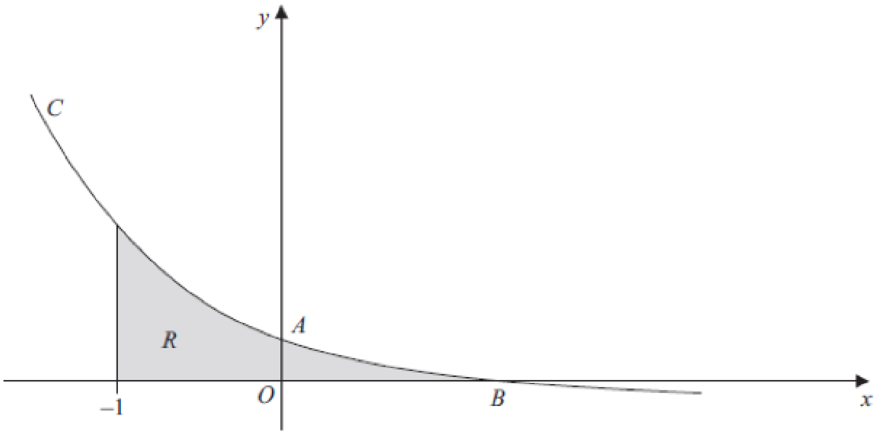


**Figure 2**

Figure 2 shows a sketch of part of the curve *C* with parametric equations

$x = 1 - \frac{1}{2}t$ ,  $y = 2^t - 1$ .

The curve crosses the $y$ -axis at the point $A$ and crosses the $x$ -axis at the point $B$ .

The point $A$ has coordinates $(0,3)$ and the point $B$ has coordinates $(1,0)$ .

The region *R*, as shown shaded in Figure 2, is bounded by the curve *C*, the line $x = -1$ and the $x$ -axis.

Use integration to find the exact area of *R*.

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## Question 34
**Categorisation: Solve a differential equation.**

*[Edexcel C4 Jan 2010 Q5b Edited]*

It is given that $\int \frac{9x+6}{x} dx = 9x + 6 \ln x + c$ , $x > 0$ .

Given that $y = 8$ at $x = 1$ , solve the differential equation

$$\frac{dy}{dx} = \frac{(9x+6)y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2 = g(x)$ .

$y^2 =$ ..........................

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## Question 35
**Categorisation: As above, but using trig functions.**

*[Edexcel C4 June 2014 Q6iii]*

Given that $y = \frac{\pi}{6}$ at $x = 0$ , solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y$$

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## Question 36

**Categorisation: As above.**

*[Edexcel C4 June 2012 Q4]* Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}$$

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## Question 37

**Categorisation: As above.**

*[Edexcel C4 June 2013 Q6a Edited]*

Water is being heated in a kettle. At time $t$ seconds, the temperature of the water is $\theta$

The rate of increase of the temperature of the water at any time $t$ is modelled by the differential equation

$\frac{d\theta}{dt} = \lambda(120 - \theta)$, $\quad \theta \leq 100$

where $\lambda$ is a positive constant.

Given that $\theta = 20$ when $t = 0$, solve this differential equation to show that

$$\theta = a - be^{-\lambda t}$$

where $a$ and $b$ are integers to be determined.

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## Question 38

**Categorisation: Answer questions about meerkats.**

*[Edexcel A2 SAM P2 Q16c Edited]* It can be shown that

$$\frac{1}{P(11-2P)} \equiv \frac{1}{11P} + \frac{2}{11(11-2P)}$$

A population of meerkats is being studied. The population is modelled by the differential equation

$\frac{dP}{dt} = \frac{1}{22}P(11-2P)\,, \quad t \geq 0\,, \quad 0 < P < 5.5$

where $P$ , in thousands, is the population of meerkats and $t$ is the time measured in years since the study began. Given that there were 1000 meerkats in the population when the study began, show that

$$P = \frac{A}{B + Ce^{-\frac{1}{2}t}}$$

where $A$ , $B$ and $C$ are integers to be found.

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## Question 39

**Categorisation: Understand the meaning of constants within a differential equation.**

*[Edexcel C4 June 2013(R) Q8a]* In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let $x$ be the mass of waste products, in kg, at time $t$ minutes after the start of the experiment. It is known that at time $t$ minutes, the rate of increase of the mass of waste products, in kg per minute, is $k$ times the mass of unburned fuel remaining, where $k$ is a positive constant.

The differential equation connecting $x$ and $t$ may be written in the form
$\frac{dx}{dt} = k(M-x)\,,$ where $M$ is a constant.

Explain, in the context of the problem, what $\frac{dx}{dt}$ and $M$ represent.

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## Question 40

**Categorisation: Integrate by parts requiring prior factorisation of the denominator.**

*[MAT 2003 1D]* What is the exact value of the definite integral

$$\int_1^2 \frac{dx}{x + x^3}$$

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## Question 41

**Categorisation: Use the double angle rule for sin to integrate.**

Determine $\int\ sin\ x\ cos\ xdx$

.......................... $+c$

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## Question 42

**Categorisation: As above.**

Determine $sin\ ^2x\ cos\ ^2xdx$

.......................... $+c$

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## Question 43
**Categorisation: Integrate by inspection involving a power of a trig function.**

Determine $\int\ sec^{3}x\ tan\ xdx$

.......................... $+c$

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## Question 44
**Categorisation: As above.**

Determine $\int\ sin^{3}x\ cos\ xdx$

.......................... $+c$

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## Question 45
**Categorisation: Use the trapezium rule to approximate the area under a curve.**

*[Edexcel A2 Specimen Papers P1 Q1b Edited]*

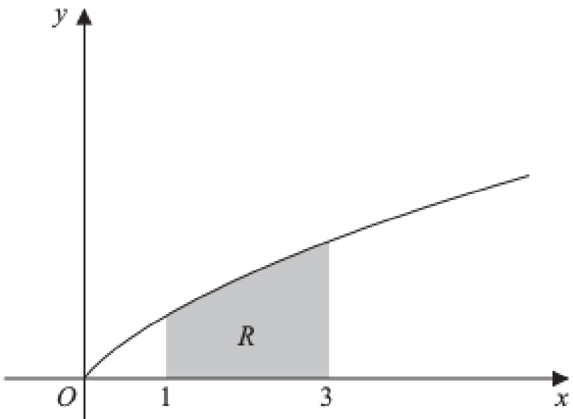


**Figure 1**

Figure 1 shows a sketch of the curve with equation $y = \frac{x}{1+\sqrt{x}}$, $x \geq 0$

The finite region $R$ , shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 1$ , the $x$ -axis and the line with equation $x = 3$

The table below shows corresponding values of $x$ and $y$ for $y = \frac{x}{1+\sqrt{x}}$

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 0.6742 | 0.8284 | 0.9686 | 1.0981 |

The trapezium rule can be used to find an estimate for the area of $R$ using the values in the table.

Explain how the trapezium rule can be used to give a better approximation for the area of $R$ .

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## Question 46

**Categorisation: Use an area approximated using the trapezium rule to find a similar area.**

*[Edexcel A2 Specimen Papers P1 Q1ci Edited]* (Continued from above)

The finite region $R$ , shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 1$ , the $x$ -axis and the line with equation $x = 3$

The table below shows corresponding values of $x$ and $y$ for $y = \frac{x}{1+\sqrt{x}}$

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 0.6742 | 0.8284 | 0.9686 | 1.0981 |

Using the trapezium rule, with all the values of $y$ in the table, an estimate for the area of $R$ is given as 1.635

Giving your answer to 3 decimal places in each case, use this answer to deduce an estimate for $\int_1^3 \frac{5x}{1+\sqrt{x}} dx$

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## Question 47

**Categorisation: As above.**

*[Edexcel A2 Specimen Papers P1 Q1cii Edited]* (Continued from above)

Figure 1 shows a sketch of the curve with equation $y = \frac{x}{1+\sqrt{x}}, \quad x \geq 0$

The finite region $R$ , shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 1$ , the $x$ -axis and the line with equation $x = 3$

The table below shows corresponding values of $x$ and $y$ for $y = \frac{x}{1+\sqrt{x}}$

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 0.6742 | 0.8284 | 0.9686 | 1.0981 |

Using the trapezium rule, with all the values of $y$ in the table, an estimate for the area of $R$ is given as 1.635

Giving your answer to 3 decimal places in each case, use this answer to deduce an estimate for $\int_1^3 \left(6 + \frac{x}{1+\sqrt{x}}\right) dx$

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## Question 48

**Categorisation: Use the trapezium rule where the step size $h$ needs to be determined.**

*[OCR C2 June 2016 Q8v]* Use the trapezium rule, with 2 strips each of width 1.5, to find an estimate for $\int_1^4 3^{x-2} dx$ .

Give your answer correct to 3 significant figures.

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# Answers

## Question 1

$3e^{2x+1}$

| | |
|---|---|
| Obtain integral of form $ke^{2x+1}$ | M1 |
| Obtain correct $3e^{2x+1}$ | A1 |

## Question 2

$-\frac{1}{3}\cos 3x + c$

## Question 3

$4x - 4\ ln\ |x| - \frac{1}{x}$

Expand to produce form $k_1 + \frac{k_2}{x} + \frac{k_3}{x^2}$

Obtain $4x - 4\ln x - \frac{1}{x}$ or $4x - 4\ln x - x^{-1}$

## Question 4

$$k = \frac{7}{2}$$

| | |
|---|---|
| Writes $\int \frac{t+1}{t}\,dt = \int 1 + \frac{1}{t}\,dt$ and attempts to integrate | M1 |
| $= t + \ln t\ (+c)$ | M1 |
| $(2a + \ln 2a) - (a + \ln a) = \ln 7$ | M1 |
| $a = \ln\frac{7}{2}$ with $k = \frac{7}{2}$ | A1 |

## Question 5

$\frac{1}{5}(2x-1)^{\frac{5}{2}}$

| | | | |
|---|---|---|---|
| (a) | $\left\{\int (2x-1)^{\frac{3}{2}}\,dx\right\} = \frac{1}{5}(2x-1)^{\frac{5}{2}}\ \{+c\}$ | $(2x \pm 1)^{\frac{3}{2}} \to \pm\lambda(2x \pm 1)^{\frac{5}{2}}$ **or** $\pm\lambda u^{\frac{5}{2}}$ where $u = 2x \pm 1$; $\lambda \neq 0$ | M1 |
| | | $\frac{1}{5}(2x-1)^{\frac{5}{2}}$ with or without $+c$. Must be simplified. | A1 |

## Question 6

$\frac{6}{5}\sqrt{5x-1}$

## Question 7

$\tan x - x$

## Question 8

$\frac{1}{3}\tan 3x$

## Question 9

$-\cot x$

## Question 10

$\frac{1}{2}x - \frac{1}{4}\sin 2x$

## Question 11

$\frac{3^x}{\ln 3}$

## Question 12

$-\frac{1}{2}e^{-x^2} + c$

## Question 13

$\frac{1}{2}\ln|x^2+1|$

## Question 14

$-2\ln y + 3\ln(3y+2)$

| | | | |
|---|---|---|---|
| (i) **Way 1** | $\frac{3y-4}{y(3y+2)} \equiv \frac{A}{y} + \frac{B}{(3y+2)} \Rightarrow 3y-4 = A(3y+2)+By$ | See notes | M1 |
| | $y=0 \Rightarrow -4=2A \Rightarrow A=-2$ | At least one of their $A=-2$ **or** their $B=9$ | A1 |
| | $y=-\frac{2}{3} \Rightarrow -6=-\frac{2}{3}B \Rightarrow B=9$ | Both their $A=-2$ **and** their $B=9$ | A1 |
| | $\int \frac{3y-4}{y(3y+2)}\,dy = \int \frac{-2}{y} + \frac{9}{(3y+2)}\,dy$ | Integrates to give at least one of **either** $\frac{A}{y} \to \pm\lambda\ln y$ **or** $\frac{B}{(3y+2)} \to \pm\mu\ln(3y+2)$, $A \neq 0, B \neq 0$ | M1 |
| | | At least one term correctly followed through from their $A$ or from their $B$ | A1 ft |
| | $= -2\ln y + 3\ln(3y+2)\ \{+c\}$ | $-2\ln y + 3\ln(3y+2)$ **or** $-2\ln y + 3\ln(y+\frac{2}{3})$ with correct bracketing, simplified or un-simplified. Can apply isw. | A1 **cao** |

## Question 15

$\int \frac{8}{(2x-1)^3} dx = -2(2x-1)^{-2}$

| | | | |
|---|---|---|---|
| (ii) | $\int \frac{8}{(2x-1)^3} dx = \frac{8(2x-1)^{-2}}{(2)(-2)} \{+c\}$ | $\pm\lambda(2x-1)^{-2}$ | M1 |
| | $\{= -2(2x-1)^{-2} \{+c\}\}$ | $\frac{8(2x-1)^{-2}}{(2)(-2)}$ or equivalent. | A1 |
| | | *{Ignore subsequent working}*. | |

## Question 16

$ln\ x - ln\ (3x-1) - \frac{1}{3x-1}$

| | | |
|---|---|---|
| (b)(i) | $\int \left(\frac{1}{x} - \frac{3}{3x-1} + \frac{3}{(3x-1)^2}\right) dx$ | |
| | $= \ln x - \frac{3}{3}\ln(3x-1) + \frac{3}{(-1)3}(3x-1)^{-1} \quad (+C)$ | M1 A1ft A1ft |
| | $\left(= \ln x - \ln(3x-1) - \frac{1}{3x-1} \quad (+C)\right)$ | |

## Question 17

$\int xe^{4x} dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$

| | | | |
|---|---|---|---|
| (i) | $\int xe^{4x}\, dx = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}\, \{dx\}$ | $\pm\alpha xe^{4x} - \int \beta e^{4x}\{dx\},\ \alpha \neq 0, \beta > 0$ | M1 |
| | | $\frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x}\, \{dx\}$ | A1 |
| | $= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \{+c\}$ | $\frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$ | A1 |

## Question 18

$\frac{\pi}{4}$

| | |
|---|---|
| $\int x\sin 2x\, dx = -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2} dx$ | M1 A1 A1 |
| $= \quad \dots \quad + \frac{\sin 2x}{4}$ | M1 |
| $[\ \dots\ ]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ | M1 A1 |

## Question 19

$p = \frac{1}{4}, q = -\frac{5}{4}$

| | | |
|---|---|---|
| $\left\{\int xe^{2x}\,dx\right\}$, $\begin{Bmatrix} u = x & \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = e^{2x} & \Rightarrow v = \frac{1}{2}e^{2x} \end{Bmatrix}$ | | |
| $\left\{\int xe^{2x}\,dx\right\} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\{dx\}$ | M1 | 3.1a |
| $\left\{\int 2e^{2x} - xe^{2x}\,dx\right\} = e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\{dx\}\right)$ | M1 | 1.1b |
| $= e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$ | A1 | 1.1b |
| $\text{Area}(R) = \int_0^2 2e^{2x} - xe^{2x}\,dx = \left[\frac{5}{4}e^{2x} - \frac{1}{2}xe^{2x}\right]_0^2$ | M1 | 2.2a |
| $= \left(\frac{5}{4}e^4 - e^4\right) - \left(\frac{5}{4}e^{2(0)} - \frac{1}{2}(0)e^0\right) = \frac{1}{4}e^4 - \frac{5}{4}$ | A1 | 2.1 |

## Question 20

$x \ln x - x + c$

## Question 21

$\frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}}$

| | |
|---|---|
| (b) $\int x^{\frac{1}{2}} \ln 2x\,dx = \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x}\,dx$ | M1 A1 |
| $= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \frac{2}{3}x^{\frac{1}{2}}\,dx$ | |
| $= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}} \quad (+C)$ | M1 A1 |

## Question 22

$S = \frac{28}{27} + \ln 27$

| | |
|---|---|
| For integration by parts on $\int x^2 \ln x\,dx$ | M1 |
| $= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$ | A1 |
| $\int -2x + 5\,dx = -x^2 + 5x \quad (+c)$ | B1 |
| All integration attempted and limits used<br>Area of $S = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5\,dx = \left[\frac{x^3}{9}\ln x - \frac{x^3}{27} - x^2 + 5x\right]_{x=1}^{x=3}$ | M1 |
| Uses correct ln laws, simplifies and writes in required form | M1 |
| Area of $S = \frac{28}{27} + \ln 27 \quad (a = 28, b = 27, c = 27)$ | A1 |

## Question 23

$x^2e^x - 2(xe^x - e^x)$

| | | |
|---|---|---|
| **(a)** $\int x^2e^x\,dx$, 1st Application: $\left\{\begin{array}{l} u = x^2 \Rightarrow \frac{du}{dx} = 2x \\ \frac{dv}{dx} = e^x \Rightarrow v = e^x \end{array}\right\}$, 2nd Application: $\left\{\begin{array}{l} u = x \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = e^x \Rightarrow v = e^x \end{array}\right\}$ | | |
| $= x^2e^x - \int 2xe^x dx$ | $x^2e^x - \int \lambda xe^x \{dx\}$, $\lambda > 0$ | M1 |
| | $x^2e^x - \int 2xe^x \{dx\}$ | A1 oe |
| $= x^2e^x - 2\left(xe^x - \int e^x dx\right)$ | **Either** $\pm Ax^2e^x \pm Bxe^x \pm C\int e^x \{dx\}$ **or** for $\pm K\int xe^x\{dx\} \to \pm K\left(xe^x - \int e^x\{dx\}\right)$ | M1 |
| $= x^2e^x - 2(xe^x - e^x)\ \{+c\}$ | $\pm Ax^2e^x \pm Bxe^x \pm Ce^x$ | M1 |
| | Correct answer, with/without $+c$ | A1 |

## Question 24

$\frac{1}{2}e^x(\sin x - \cos x)$

## Question 25

$\frac{11}{3} + 2\ln 2 - 2\ln 3$

| | | |
|---|---|---|
| (c) $\{u = 1 + \sqrt{x}\} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{dx}{du} = 2(u-1)$ | | B1 |
| $\left\{\int \frac{x}{1+\sqrt{x}}\,dx = \right\} \int \frac{(u-1)^2}{u}.\,2(u-1)\ du$ | $\int \frac{(u-1)^2}{u}$...... | M1 |
| | $\int \frac{(u-1)^2}{u}.\,2(u-1)$ | A1 |
| $= 2\int \frac{(u-1)^3}{u}\,du = \{2\}\int \frac{(u^3 - 3u^2 + 3u - 1)}{u}\,du$ | Expands to give a "four term" cubic in $u$. Eg: $\pm Au^3 \pm Bu^2 \pm Cu \pm D$ | M1 |
| $= \{2\}\int\left(u^2 - 3u + 3 - \frac{1}{u}\right)du$ | An attempt to divide at least three terms in ***their cubic*** by $u$. See notes. | M1 |
| $= \{2\}\left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$ | $\int \frac{(u-1)^3}{u} \to \left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$ | A1 |
| $\text{Area}(R) = \left[\frac{2u^3}{3} - 3u^2 + 6u - 2\ln u\right]_2^3$ | | |
| $= \left(\frac{2(3)^3}{3} - 3(3)^2 + 6(3) - 2\ln 3\right) - \left(\frac{2(2)^3}{3} - 3(2)^2 + 6(2) - 2\ln 2\right)$ | Applies limits of 3 and 2 in $u$ or 4 and 1 in $x$ and subtracts either way round. | M1 |
| $= \frac{11}{3} + 2\ln 2 - 2\ln 3$ or $\frac{11}{3} + 2\ln\left(\frac{2}{3}\right)$ or $\frac{11}{3} - \ln\left(\frac{9}{4}\right)$, etc | Correct exact answer or equivalent. | A1 |

## Question 26

$e(e-1)$

| | | |
|---|---|---|
| $\frac{du}{dx} = -\sin x$ | | B1 |
| $\int \sin x\, e^{\cos x+1}dx = -\int e^u du$ | | M1 A1 |
| $= -e^u$ <br> $= -e^{\cos x+1}$ | ft sign error | A1ft |
| $\left[-e^{\cos x+1}\right]_0^{\frac{\pi}{2}} = -e^1 - \left(-e^2\right)$ | or equivalent with $u$ | M1 |
| $= e(e-1)$ ✱ | cso | A1 |

## Question 27

$$a = -2$$

| | | |
|---|---|---|
| $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,d\theta$ | | |
| Attempts this question by applying the substitution $u = 1+\cos\theta$ and progresses as far as achieving $\int \ldots \frac{(u-1)}{u} \ldots$ | M1 | 3.1a |
| $u = 1+\cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$ | M1 | 1.1b |
| $\left\{\int \frac{\sin 2\theta}{1+\cos\theta}\,d\theta = \right\} \int \frac{2\sin\theta\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{-2(u-1)}{u}\,du$ | A1 | 2.1 |
| $-2\int\left(1-\frac{1}{u}\right)du = -2(u - \ln u)$ | M1 | 1.1b |
| | M1 | 1.1b |
| $\left\{\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,d\theta = \right\} = -2\left[u - \ln u\right]_2^1 = -2((1-\ln 1)-(2-\ln 2))$ | M1 | 1.1b |
| $= -2(-1+\ln 2) = 2 - 2\ln 2$ * | A1* | 2.1 |

## Question 28

$\lambda = 8$

| | | | |
|---|---|---|---|
| **(ii) (a) Way 1** | $\{x = 4\sin^2\theta \Rightarrow\}\ \frac{dx}{d\theta} = 8\sin\theta\cos\theta$ or $\frac{dx}{d\theta} = 4\sin 2\theta$ or $dx = 8\sin\theta\cos\theta d\theta$ | | B1 |
| | $\int\sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}.8\sin\theta\cos\theta\,\{d\theta\}$ or $\int\sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}.4\sin 2\theta\,\{d\theta\}$ | | M1 |
| | $= \int \underline{\underline{\tan\theta}}.8\sin\theta\cos\theta\,\{d\theta\}$ or $\int \underline{\underline{\tan\theta}}.4\sin 2\theta\,\{d\theta\}$ | $\sqrt{\left(\frac{x}{4-x}\right)} \to \pm K\tan\theta$ or $\pm K\left(\frac{\sin\theta}{\cos\theta}\right)$ | <u>M1</u> |
| | $= \int 8\sin^2\theta\,d\theta$ | $\int 8\sin^2\theta\,d\theta$ including $d\theta$ | A1 |
| | $3 = 4\sin^2\theta$ **or** $\frac{3}{4} = \sin^2\theta$ **or** $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$ $\{x = 0 \to \theta = 0\}$ | Writes down a correct equation involving $x = 3$ leading to $\theta = \frac{\pi}{3}$ and no incorrect work seen regarding limits | B1 |

## Question 29

$\frac{\sqrt{3}-1}{4}$

## Question 30

$2 + 2\,ln\left(\frac{3}{5}\right)$

| | | |
|---|---|---|
| $\int_0^4 \frac{1}{2+\sqrt{(2x+1)}}\,dx\ ,\ u = 2+\sqrt{(2x+1)}$ | | |
| $\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = u-2$ | Either $\frac{du}{dx} = \pm K(2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = \pm\lambda(u-2)$ | M1 |
| | Either $\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = (u-2)$ | A1 |
| $\left\{\int \frac{1}{2+\sqrt{(2x+1)}}\,dx\right\} = \int \frac{1}{u}(u-2)\,du$ | Correct substitution (Ignore integral sign and $du$). | A1 |
| $= \int \left(1-\frac{2}{u}\right)du$ | An attempt to divide each term by $u$. | dM1 |
| | $\pm Au \pm B\ln u$ | ddM1 |
| $= u - 2\ln u$ | $u - 2\ln u$ | A1 ft |
| $\{\text{So } [u-2\ln u]_3^5\} = (5-2\ln 5)-(3-2\ln 3)$ | Applies limits of 5 and 3 in $u$ or 4 and 0 in $x$ in their integrated function and subtracts the correct way round. | M1 |
| $= 2+2\ln\left(\frac{3}{5}\right)$ | $2+2\ln\left(\frac{3}{5}\right)$ | A1 **cao cso** |

## Question 31

$ln\ \left(\frac{3}{2}\right)$

| | | |
|---|---|---|
| Attempts to apply $\int y\frac{dx}{dt}\,dt$ | M1 | 3.1a |
| $\left\{\int y\frac{dx}{dt}\,dt = \right\} = \int \left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right)dt$ | A1 | 1.1b |
| $\frac{1}{(t+1)(t+2)} \equiv \frac{A}{(t+1)} + \frac{B}{(t+2)} \Rightarrow 1 \equiv A(t+2)+B(t+1)$ | M1 | 3.1a |
| $\{A=1, B=-1 \Rightarrow\}$ gives $\frac{1}{(t+1)} - \frac{1}{(t+2)}$ | A1 | 1.1b |
| $\left\{\int\left(\frac{1}{(t+1)} - \frac{1}{(t+2)}\right)dt = \right\}\ \ln(t+1)-\ln(t+2)$ | M1 | 1.1b |
| | A1 | 1.1b |
| $\text{Area}(R) = [\ln(t+1)-\ln(t+2)]_0^2 = (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$ | M1 | 2.2a |
| $= \ln 3 - \ln 4 + \ln 2 = \ln\left(\frac{(3)(2)}{4}\right) = \ln\left(\frac{6}{4}\right)$ | | |
| $= \ln\left(\frac{3}{2}\right)$ * | A1* | 2.1 |

## Question 32

648

| | | |
|---|---|---|
| (b) $\frac{dx}{dt} = 10t$ | Seen or implied | B1 |
| $\int y\,dx = \int y\frac{dx}{dt}\,dt = \int t(9-t^2)10t\,dt$ | | M1 A1 |
| $= \int(90t^2 - 10t^4)\,dt$ | | |
| $= \frac{90t^3}{3} - \frac{10t^5}{5}\ (+C) \quad (= 30t^3 - 2t^5\ (+C))$ | | A1 |
| $\left[\frac{90t^3}{3} - \frac{10t^5}{5}\right]_0^3 = 30\times3^3 - 2\times3^5 \quad (=324)$ | | M1 |
| $A = 2\int y\,dx = 648 \quad (\text{units}^2)$ | | A1 |

# Question 33

$\frac{15}{2\ ln\ 2} - 2$

| | | | |
|---|---|---|---|
| (d) | $\text{Area}(R) = \int (2^t - 1).\left(-\frac{1}{2}\right)\mathrm{d}t$ | Complete substitution for both $y$ and $\mathrm{d}x$ | M1 |
| | $x = -1 \rightarrow t = 4$ and $x = 1 \rightarrow t = 0$ | | B1 |
| | $= \left\{-\frac{1}{2}\right\}\left(\frac{2^t}{\ln 2} - t\right)$ | Either $2^t \rightarrow \frac{2^t}{\ln 2}$<br>or $(2^t - 1) \rightarrow \frac{(2^t)}{\pm\alpha(\ln 2)} - t$<br>or $(2^t - 1) \rightarrow \pm\alpha(\ln 2)(2^t) - t$ | M1* |
| | | $(2^t - 1) \rightarrow \frac{2^t}{\ln 2} - t$ | A1 |
| | $\left\{-\frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_4^0\right\} = -\frac{1}{2}\left(\left(\frac{1}{\ln 2}\right) - \left(\frac{16}{\ln 2} - 4\right)\right)$ | **Depends on the previous method mark.** Substitutes their changed limits in $t$ and subtracts either way round. | dM1* |
| | $= \frac{15}{2\ln 2} - 2$ | $\frac{15}{2\ln 2} - 2$ or equivalent. | A1 |

# Question 34

$y^2 = (6x + 4\ ln\ x - 2)^3$

| | | | |
|---|---|---|---|
| (b) | $\int \frac{1}{y^{\frac{1}{3}}}\mathrm{d}y = \int \frac{9x+6}{x}\mathrm{d}x$ | Integral signs not necessary | B1 |
| | $\int y^{-\frac{1}{3}}\mathrm{d}y = \int \frac{9x+6}{x}\mathrm{d}x$ | | |
| | $\frac{y^{\frac{2}{3}}}{\frac{2}{3}} = 9x + 6\ln x\ (+C)$ | $\pm ky^{\frac{2}{3}} =$ their (a) | M1 |
| | $\frac{3}{2}y^{\frac{2}{3}} = 9x + 6\ln x\ (+C)$ | ft their (a) | A1ft |
| $y = 8,\ x = 1$ | | | |
| | $\frac{3}{2}8^{\frac{2}{3}} = 9 + 6\ln 1 + C$ | | M1 |
| | $C = -3$ | | A1 |
| | $y^{\frac{2}{3}} = \frac{2}{3}(9x + 6\ln x - 3)$ | | |
| | $y^2 = (6x + 4\ln x - 2)^3 \quad \left(= 8(3x + 2\ln x - 1)^3\right)$ | | A1 |

# Question 35

$\frac{2}{3}\ sin\ ^3y = e^x - \frac{11}{12}$

| | | |
|---|---|---|
| (iii) | $\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y \qquad y = \frac{\pi}{6}$ at $x = 0$ | |
| | **Main Scheme** | |
| | $\int \frac{1}{\operatorname{cosec} 2y \operatorname{cosec} y} dy = \int e^x dx$ **or** $\int \sin 2y \sin y \, dy = \int e^x dx$ | B1 oe |
| | $\int 2\sin y \cos y \sin y \, dy = \int e^x dx$ Applying $\frac{1}{\operatorname{cosec} 2y}$ or $\sin 2y \to 2\sin y \cos y$ | M1 |
| | Integrates to give $\pm \mu \sin^3 y$ | M1 |
| | $\frac{2}{3}\sin^3 y = e^x \{+c\}$ $\qquad 2\sin^2 y \cos y \to \frac{2}{3}\sin^3 y$ | A1 |
| | $e^x \to e^x$ | B1 |
| | $\frac{2}{3}\sin^3\left(\frac{\pi}{6}\right) = e^0 + c$ **or** $\frac{2}{3}\left(\frac{1}{8}\right) - 1 = c$ $\qquad$ Use of $y = \frac{\pi}{6}$ and $x = 0$ in an integrated equation containing $c$ | M1 |
| | $\left\{\Rightarrow c = -\frac{11}{12}\right\}$ giving $\frac{2}{3}\sin^3 y = e^x - \frac{11}{12}$ $\qquad \frac{2}{3}\sin^3 y = e^x - \frac{11}{12}$ | A1 |

# Question 36

$\frac{1}{2}y^2 = 3\tan x - 1$

| | | |
|---|---|---|
| $\int y\,dy = \int \frac{3}{\cos^2 x} dx$<br>$= \int 3\sec^2 x \, dx$ | Can be implied. Ignore integral signs | B1 |
| $\frac{1}{2}y^2 = 3\tan x \quad (+C)$ | | M1 A1 |
| $y = 2, x = \frac{\pi}{4}$<br>$\frac{1}{2}2^2 = 3\tan\frac{\pi}{4} + C$ | | M1 |
| Leading to<br>$C = -1$<br>$\frac{1}{2}y^2 = 3\tan x - 1$ | or equivalent | A1 |

# Question 37

$a = 120$ , $b = 100$

| | | | |
|---|---|---|---|
| **(a)** | $\int \frac{1}{120-\theta} d\theta = \int \lambda \, dt$ or $\int \frac{1}{\lambda(120-\theta)} d\theta = \int dt$ | | B1 |
| | $-\ln(120-\theta); = \lambda t + c$ or $-\frac{1}{\lambda}\ln(120-\theta); = t + c$ | See notes | M1 A1;<br>M1 A1 |
| | $\{t = 0, \theta = 20 \Rightarrow\} -\ln(120-20) = \lambda(0) + c$ | See notes | M1 |
| | $c = -\ln 100 \Rightarrow -\ln(120-\theta) = \lambda t - \ln 100$ | | |
| | ***then either...*** $-\lambda t = \ln(120-\theta) - \ln 100$<br>$-\lambda t = \ln\left(\frac{120-\theta}{100}\right)$<br>$e^{-\lambda t} = \frac{120-\theta}{100}$ | ***or...*** $\lambda t = \ln 100 - \ln(120-\theta)$<br>$\lambda t = \ln\left(\frac{100}{120-\theta}\right)$<br>$e^{\lambda t} = \frac{100}{120-\theta}$ | dddM1 |
| | $100e^{-\lambda t} = 120 - \theta$ | $(120-\theta)e^{\lambda t} = 100$<br>$\Rightarrow 120 - \theta = 100e^{-\lambda t}$ | |
| | leading to $\theta = 120 - 100e^{-\lambda t}$ | | A1 * |

## Question 38

$A = 11\,, B = 2\,, C = 9$

| | | |
|---|---|---|
| Uses ln laws | $2\ln P - 2\ln(11-2P) = t - 2\ln 9$ <br> $\Rightarrow \ln\left(\dfrac{9P}{11-2P}\right) = \dfrac{1}{2}t$ | M1 |
| Makes '$P$' the subject | $\Rightarrow \left(\dfrac{9P}{11-2P}\right) = e^{\frac{1}{2}t}$ <br> $\Rightarrow 9P = (11-2P)e^{\frac{1}{2}t}$ <br> $\Rightarrow P = f\left(e^{\frac{1}{2}t}\right)$ or $\Rightarrow P = f\left(e^{-\frac{1}{2}t}\right)$ | M1 |
| | $\Rightarrow P = \dfrac{11}{2+9e^{-\frac{1}{2}t}} \Rightarrow A = 11, B = 2, C = 9$ | A1 |

## Question 39

"rate of increase OR rate of change" and "mass"

| | | | |
|---|---|---|---|
| **(a)** | $\dfrac{dx}{dt}$ is the rate of increase of the mass of waste products. | Any one correct explanation. | B1 |
| | $M$ is the total mass of unburned fuel and waste fuel (or the initial mass of unburned fuel) | Both explanations are correct. | B1 |

## Question 40

$\frac{1}{2}\ ln\ \frac{8}{5}$

## Question 41

$-\frac{1}{4}\ cos\ 2x$

$sin\ x\ cos\ x$ can be written as $\frac{1}{2}\ sin\ 2x$ .

## Question 42

$\frac{1}{8}x - \frac{1}{32}\ sin\ 4x$

$$sin^2 x\ cos^2 x = (\ sin\ x\ cos\ x)^2$$

$$= \left(\frac{1}{2}\ sin\ 2x\right)^2 = \frac{1}{4}\ sin^2 2x$$

$$= \frac{1}{8} - \frac{1}{8}\ cos\ 4x$$

## Question 43

$\frac{1}{3}\sec^3 x$

## Question 44

$\frac{1}{4}\sin^4 x$

## Question 45

"more OR increase OR decrease the width OR narrower OR smaller"

| | | |
|---|---|---|
| Any valid reason, for example<br>• Increase the number of strips<br>• Decrease the width of the strips<br>• Use more trapezia between $x = 1$ and $x = 3$ | B1 | 2.4 |

## Question 46

8.175

| | | |
|---|---|---|
| $\left\{\int_1^3 \frac{5x}{1+\sqrt{x}}\,dx\right\} = 5(\text{"}1.635\text{"}) = 8.175$ | B1ft | 2.2a |

## Question 47

13.635

| | | |
|---|---|---|
| $\left\{\int_1^3 \left(6 + \frac{x}{1+\sqrt{x}}\right)dx\right\} = 6(2) + (\text{"}1.635\text{"}) = 13.635$ | B1ft | 2.2a |

## Question 48

9.6

| | | |
|---|---|---|
| $0.5\times1.5\times\{3^{-1} + 2\times3^{0.5} + 3^2\}$<br>$= 9.60$ | B1 | State the 3 correct $y$-values, and no others |
| | M1 | Attempt use of correct trapezium rule to attempt area between $x = 1$ and $x = 4$ |
| | A1 | Obtain 9.60, or better (allow 9.6) |